Diffraction Gratings

Astronomy 525

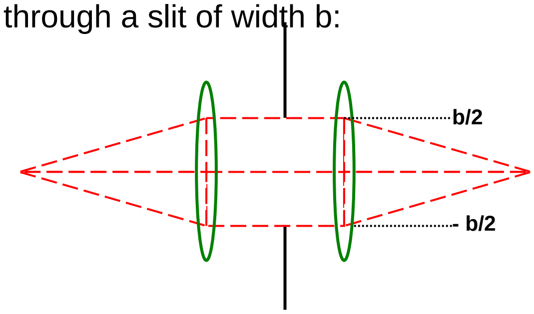
Lecture 08

Outline

- Basic principles – slit interference patterns
- The grating equation
- Resolving power
- Blazed gratings
- Alternative form of the grating equation
- Types of grating spectrometers
- Limits to resolution
- System design
- Summary

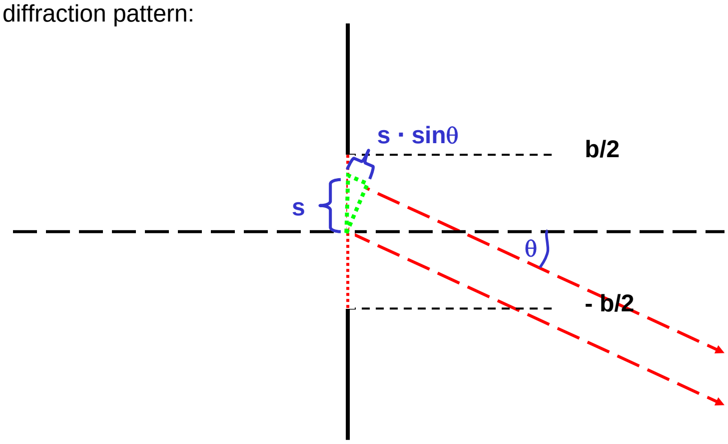
Gratings

A grating achieves its resolving power by the interference of diffracted light. To derive the response function of a grating, we first look at the image formed when a collimated beam of light passes through a slit of width b:



Single Slit -1

If the slit width is of the same order of magnitude as the wavelength, then a very prominent diffraction pattern forms the familiar Fraunhofer diffraction pattern:



Single Slit - 2

Analyzing a single slit of width b, with incident and diffracted wavefronts strictly parallel:

Intensity of the beam: $I = I_0 \left\{ \frac{\sin \beta}{\beta} \right\}^2$

With: $\beta \equiv \frac{\pi}{\lambda} b \sin \theta$

Nulls occur at: $\beta = n\pi \Leftrightarrow \sin \theta = \frac{n\lambda}{b}$

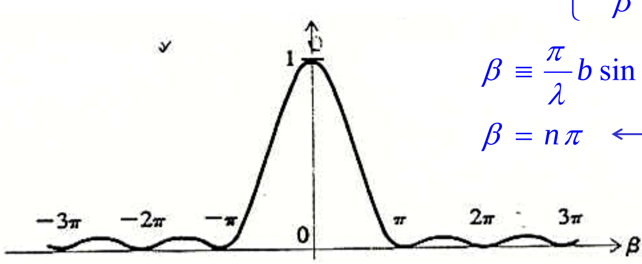
Single Slit - 3

For a given wavelength, the width of the diffraction pattern varies inversely with the slit width. For a very narrow slit, the pattern is broad, but dim. It shrinks, and becomes brighter with wider slits

$I = I_0 \left\{ \frac{\sin \beta}{\beta} \right\}^2$

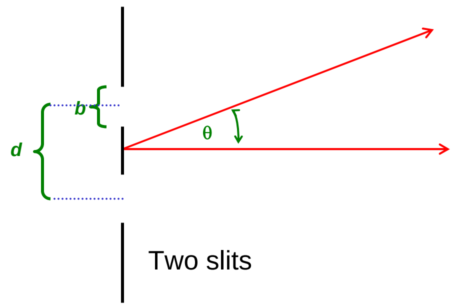
$\beta \equiv \frac{\pi}{\lambda} b \sin \theta$

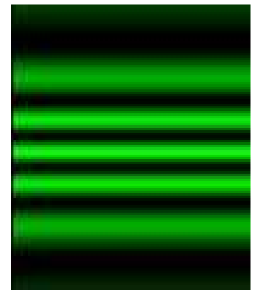
$\beta = n\pi \longleftarrow$ for minima



Two Slit Interference - 1

Now, suppose the collimated beam instead falls on two slits of width *b*, separated by a distance *d*:





Then we end up with the pattern at right: a series of bright interference fringes modulated by the envelope of the single slit.

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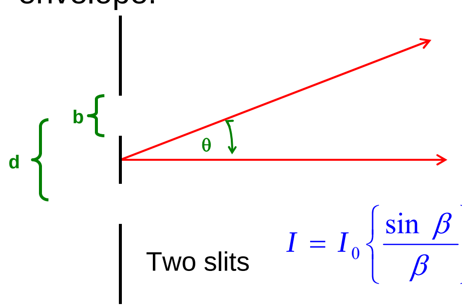
Two Slit Interference - 2

Mathematically, one can show that the bright fringes are regularly spaced, and modulated by the single slit envelope:



Same as for single slit – the envelope

$$I = I_0 \left\{ \frac{\sin \beta}{\beta} \right\}^2 \cos^2 \gamma$$

Interference term

$$\beta \equiv \frac{\pi}{\lambda} b \sin \theta$$

$$\gamma = \frac{\pi}{\lambda} d \sin \theta$$

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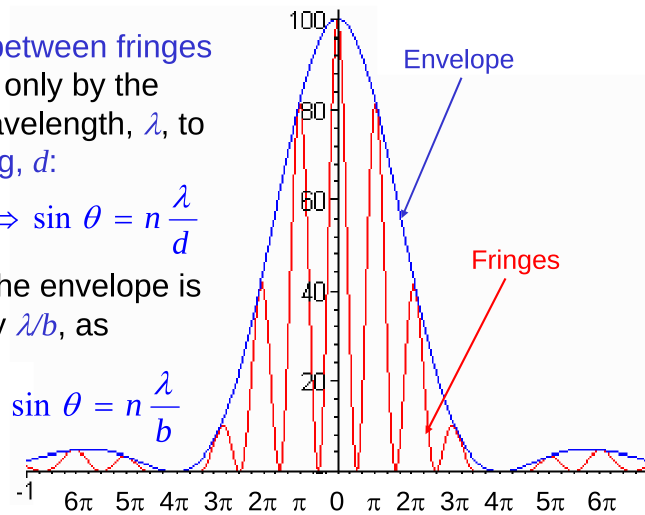
Two Slit Interference - 3

The spacing between fringes is determined only by the ratio of the wavelength, λ, to the slit spacing, d:

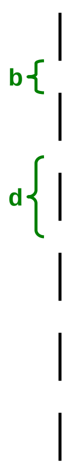
$$\gamma = \pm n\pi \Leftrightarrow \sin\theta = n\frac{\lambda}{d}$$

The width of the envelope is determined by λ/b, as before:

$$\beta = n\pi \Leftrightarrow \sin\theta = n\frac{\lambda}{b}$$



The Diffraction Grating - 1



Now suppose we have N slits, of equal opening, b, spaced by equal spacings, d. One can then show that the intensity transmission function is:

$$I = I_0 \left\{\frac{\sin\beta}{\beta}\right\}^2 \left\{\frac{\sin N\gamma}{N\sin\gamma}\right\}^2$$

number of slits, or rulings

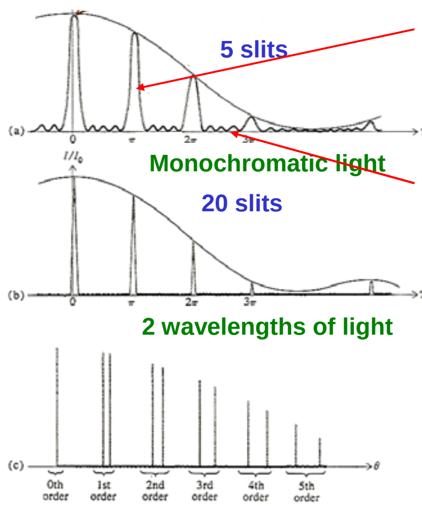
$$\beta \equiv \frac{\pi}{\lambda} b \sin\theta$$

Interference term

$$\gamma = \frac{\pi}{\lambda} d \sin\theta$$

single slit envelope

The Diffraction Grating - 2



Principle fringes occur at $\gamma = n\pi$:

$$n\lambda = d \sin\theta$$

n = diffraction "order"

Secondary fringes occur at:

$$\gamma = \frac{3\pi}{2N}, \frac{5\pi}{2N}, \frac{7\pi}{2N}, \ldots$$

The spacing between secondary fringes is given by:

$$\Delta\gamma = \frac{\pi}{N}$$

If the slits are narrow, then:

$$\frac{\sin\beta}{\beta} \approx 1$$

and the first few orders have about the same intensity.

Resolving Power of a Grating - 1

The resolving power of a grating is obtained by examining the angular width of a principle fringe, i.e., the angular distance between the peak and adjacent minimum at a fixed wavelength. This width is obtained by setting $\Delta\gamma = \frac{\pi}{N}$

Now: $\gamma = \frac{\pi}{\lambda} d \sin\theta$

$$\Rightarrow \Delta\gamma = \frac{\pi}{\lambda} d \cos\theta \cdot \Delta\theta = \frac{\pi}{N}$$

$$\Rightarrow \Delta\theta = \frac{\lambda}{Nd\cos\theta}$$

So that if N is large, $\Delta\theta$ is small, meaning narrow lines and high resolving power.

Resolving Power of a Grating - 2

For a given order, n, since $n\lambda = d\sin\theta$, the dependence of θ on λ is given by:

$$\Delta\theta = \frac{n\Delta\lambda}{d\cos\theta}$$

notice for $n = 0$, $\frac{\Delta\theta}{\Delta\lambda} = 0$!

Therefore, $\Delta\theta$ is the angular separation of lines differing in wavelength by $\Delta\lambda$. The resolving power, R, of the grating is therefore given by:

$$R \equiv \frac{\lambda}{\Delta\lambda} = Nn$$

Number of grooves

interference order

Notice for such a grating, the power falls into many orders, so the efficiency in any given order, n, is small.

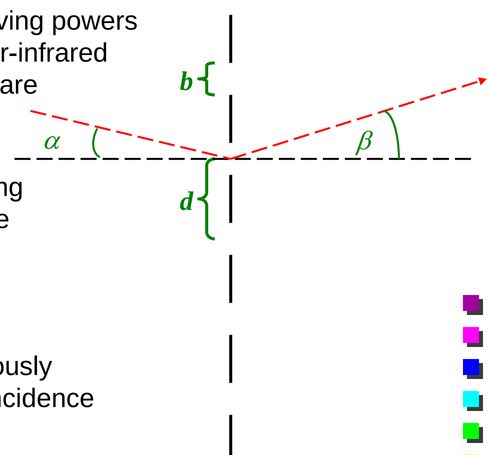
Diffraction Gratings for Astronomy

- Diffraction gratings are quite useful for achieving moderately high resolving powers in astronomy. In particular, at far-infrared wavelengths, reflective gratings are commonly used.
- For non-normal incidence, the relationship between the incoming and outgoing rays is given by the grating equation:

$$\frac{n\lambda}{d} = \sin\alpha + \sin\beta$$

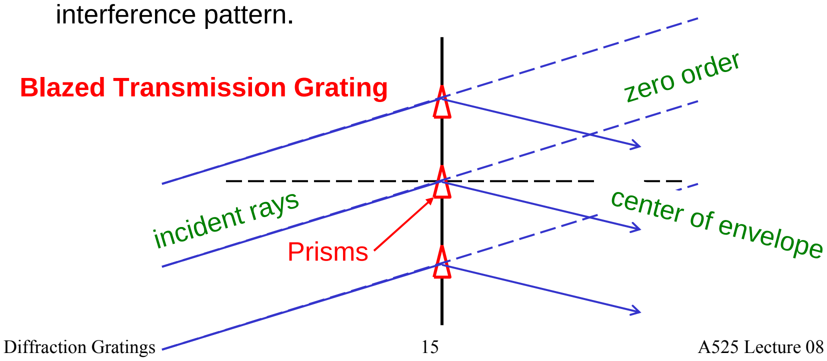
- Which is the same as that previously derived except for non-normal incidence

$$\frac{n\lambda}{d} = \sin\theta$$



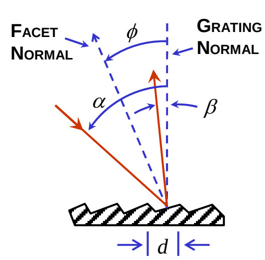
Blazed Gratings

- A problem with simple gratings is that the diffraction envelope is broadest at $\theta = 0$, where the chromatic dispersion is zero. That is most of the power is dumped into the zero order!
- This problem is solved by *"blazing"*, shifting the diffraction envelope relative to the interference pattern to dump the power into a more useful angle, such as 1st (or higher) order of the interference pattern.



Blazed Reflection Gratings - 1

- For a reflection grating, the facets are tilted by the blaze angle, ϕ, and the maximum then occurs for: $\alpha + \beta = 2\phi$ when we have specular reflection. This is termed the "blaze wavelength"
- The manufacturer normally defines the blaze angle when the grating is used in "Littrow mode", i.e. $\alpha = \beta$:



$$\frac{n\lambda_o}{d} = \sin\alpha + \sin\beta$$
$$= 2\sin\phi$$

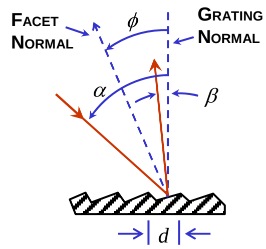
Blazed Reflection Gratings - 2

- In practice, it is difficult to use the grating in true Littrow mode since the incoming and outgoing beams are then coincident. The true blaze will appear at a shorter λ if the grating is used at some orientation where $\alpha \neq \beta$:
- Let:
  - β_B = diffraction angle where the blaze is directed
  - λ_0 = blaze wavelength for $\alpha = \beta$
  - λ_B = wavelength of true blaze
- Now: (specular reflection)

$$\alpha_B - \phi = \phi - \beta_B$$

or:

$$\alpha_B + \beta_B = 2\phi$$



Blazed Reflection Gratings - 3

- For the general case: $$\frac{n\lambda_B}{d} = \sin\alpha_B + \sin\beta_B$$
- So that combining with the equation for true Littrow mode:

$$\frac{n\lambda_0}{d} = \sin\alpha + \sin\beta = 2\sin\phi$$

We have:

$$\frac{\lambda_B}{\lambda_0} = \frac{\sin\alpha_B + \sin\beta_B}{2\sin\phi}$$

$$= \frac{2\sin\frac{1}{2}(\alpha_B + \beta_B)\cos\frac{1}{2}(\alpha_B - \beta_B)}{2\sin\phi}$$

true blaze λ

or:

$$\frac{\lambda_B}{\lambda_0} = \cos\tfrac{1}{2}\underbrace{(\alpha_B - \beta_B)} \leq 1 \qquad (\alpha_B \neq \beta_B)$$

Littrow mode blaze λ

angle between incoming and outgoing beams – ***"angular deviation"***

Blazed Reflection Gratings - 4

- Example:

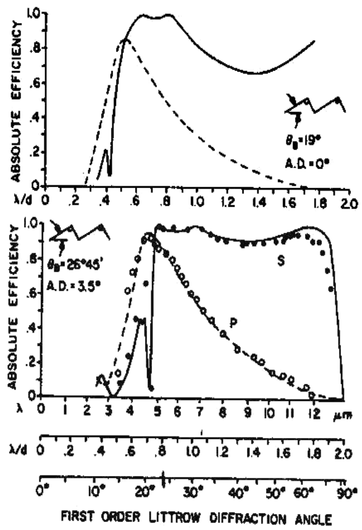
Suppose we order a grating with a blaze wavelength of 800 nm, and plan to use it with an angular deviation of 40°. Then the ratio of the Littrow mode to true blaze wavelength is:

$$\frac{\lambda_B}{\lambda_0} = \cos \tfrac{1}{2}(\alpha_B - \beta_B) = \cos \tfrac{1}{2}(40) = 0.9397$$

So that a grating that is said to be blazed at 800 nm will actually have a true blaze wavelength of:

$$\lambda_B = 0.9397\,\lambda_0 = 751.8\,\text{nm}$$

Blazed Grating Efficiency - 1



Grating efficiencies have been calculated for a variety of blaze angles (cf. E.G. Loewen, Electro-optical systems design, 1977)

-- Gratings can be very efficient over a broad band!

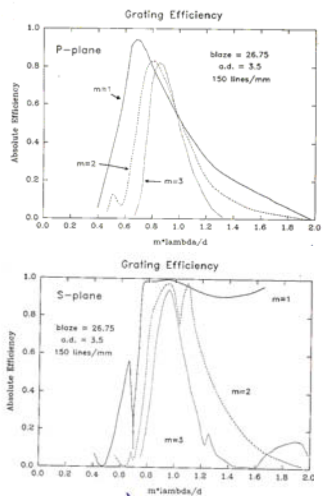
-- For example, the 26.75° blaze angle grating is more than 50% over the entire order.

-- Near blaze angle efficiency is ~ 80%.

-- Gratings are strongly polarized in general.

{P-plane denotes E field Parallel to grooves, while S-plane denotes E field perpendicular to the grating groove direction}

Blazed Grating Efficiency - 2



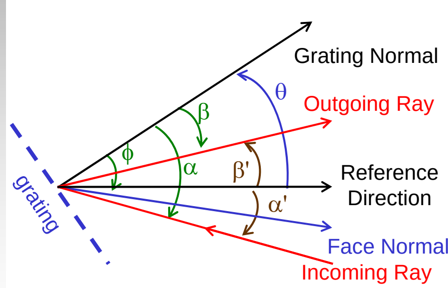
Grating efficiencies are also a function of the order in which a grating is used.

-- Efficiencies are high over a broader range for lower orders of the grating

-- Differences between P and S plane efficiencies are less pronounced in higher order gratings.

Alternative Form of the Grating Equation

- For many systems, the absolute direction of the incoming and outgoing beams are held fixed, while the resonant wavelength is adjusted by tilting the grating. For such systems, one can write a more convenient form of the grating equation:
- From the grating equation:



$$\frac{n\lambda}{d} = \sin\alpha + \sin\beta$$

By inspection:

$$\beta = \beta' - \theta; \alpha = \alpha' - \theta$$

So that:

$$\frac{n\lambda}{d} = \sin(\alpha' - \theta) + \sin(\beta' - \theta)$$

Alternative form of the Grating Equation

- Now define a zero order angle ($n = 0$) as $\theta = \theta_o$
- From the grating equation: $\alpha = -\beta$

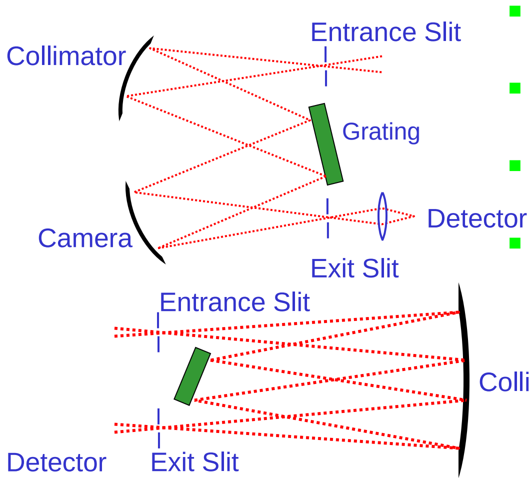
$$\Rightarrow \beta' - \theta_0 = \theta_0 - \alpha', \text{or:} \quad \theta_0 = \frac{\alpha' + \beta'}{2}$$

$$\frac{n\lambda}{d} = 2\sin\tfrac{1}{2}(\alpha' - \theta + \beta' - \theta)\cos\tfrac{1}{2}(\alpha' - \theta - \beta' + \theta)$$

$$\frac{n\lambda}{d} = 2\sin(\theta_0 - \theta)\cos\underbrace{\left(\frac{\alpha' - \beta'}{2}\right)}_{\text{fixed}}$$

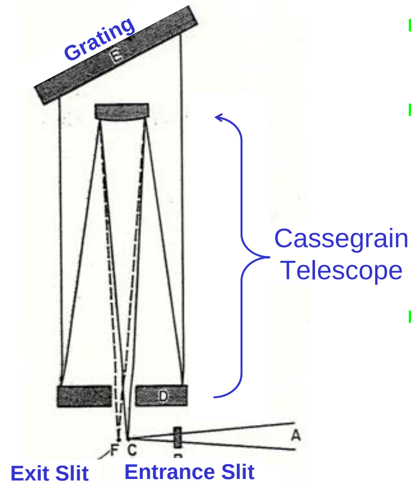
- So that the resonant wavelength is a function of tilt angle, θ

Types of Grating Spectrometers - 1



- Czerny-Turner spectrograph
- Ebert-Fastie spectrograph
- Both of these designs cancel coma
- The Czerny-Turner spectrograph is especially easy to baffle

Types of Grating Spectrometers - 2



- Simple Littrow Mode Spectrometer
- Little optical aberration for on-axis system
  Difficult to adapt to large format arrays
- Difficult to properly baffle

Slit Magnification

Let W', W = width of the entrance (exit) slit
f_{coll}, f_{cam} = collimator (camera) focal length

Then, the angular size of the entrance (exit) slit is:

$$d\alpha = \frac{W'}{f_{coll}} \qquad d\beta = \frac{W}{f_{cam}}$$

From the grating equation, the angular size of a monochromatic image of the entrance slit is:

$$d\beta = -\frac{\cos\alpha}{\cos\beta}\, d\alpha$$

Combining these three equations we have: $\boxed{W = -\frac{\cos\alpha}{\cos\beta}\frac{f_{cam}}{f_{coll}} W'}$

Which is the projected size of the entrance slit onto the exit slit, or detector

Spectral Purity

The image width, W corresponds to a wavelength interval:

$$\Delta\lambda_s = W \frac{\mathrm{d}\lambda}{\mathrm{d}x}$$

where $\mathrm{d}\lambda/\mathrm{d}x$ is the linear dispersion of the wavelengths on the detector

$$\frac{\mathrm{d}\lambda}{\mathrm{d}x} = \frac{1}{f_{cam}}\left(\frac{\mathrm{d}\beta}{\mathrm{d}\lambda}\right)^{-1}$$

the angular dispersion $\mathrm{d}\beta/\mathrm{d}\lambda$ is obtained by differentiating the grating equation at fixed α

$$\frac{\mathrm{d}\beta}{\mathrm{d}\lambda} = \frac{n}{d \cos\beta}$$

Combining the above 3 equations, we obtain the expression for the *spectral purity of the system*:

$$\boxed{\Delta\lambda_s = -\cos\alpha \frac{W'}{f_{coll}}\frac{d}{n}}$$

(where we used the derived expression for the slit magnification)

Chromatic Resolution

- The chromatic resolution is the inherent ability to resolve adjacent spectral lines, and is limited by the overall size of the diffraction grating and is given by:

  $$\boxed{\Delta\lambda_c = \frac{\lambda}{G}\frac{d}{n} = \frac{\lambda}{nN}}$$

  where G is the illuminated size of the grating, n is the grating order, and $N = G/d$ is the number of grooves (facets) in the grating

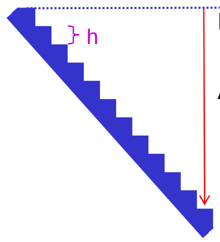
- To qualitatively understand this, result consider the Rayleigh criterion as set by the grating (or collimating mirror): $\mathrm{d}\beta \approx \lambda/G$

  But, from the grating equation: $$\frac{n\cdot\mathrm{d}\lambda}{d} = -\cos\beta\cdot\mathrm{d}\beta$$

  Combining the two results we get: $$\Delta\lambda \equiv \mathrm{d}\lambda = -\cos\beta\frac{\lambda}{G}\frac{d}{n}$$

  Which is close to the result presented above.

Chromatic Resolution: Intuitive Derivation



Resonant condition: $\mathrm{Nh} = \frac{m\lambda_0}{2}$ (1) (round trip)

Anti-resonant condition $\mathrm{Nh} = (m + \tfrac{1}{2})\frac{\lambda_S}{2}$ (2) $\lambda_S \equiv \lambda_{Short}$

$\mathrm{Nh} = (m - \tfrac{1}{2})\frac{\lambda_L}{2}$ (3) $\lambda_L \equiv \lambda_{Long}$

$$(1)\ \&\ (2) \Rightarrow m(\lambda_0 - \lambda_S) = \lambda_S/2$$
$$(1)\ \&\ (3) \Rightarrow m(\lambda_0 - \lambda_L) = -\lambda_L/2$$

-- n[th] order Littrow grating

-- N = # of grooves

-- h = groove height = n·λ/2

$$m(\lambda_L - \lambda_S) \equiv m \cdot \Delta\lambda = \frac{\lambda_L + \lambda_S}{2} = \lambda_0$$

$\frac{\lambda_0}{\Delta\lambda} = m \equiv$ Resolving Power, ***R***

Notice $m = N \cdot n$ $\left(= N \frac{2 \cdot h}{\lambda} \right)$

So that: $\boxed{\Delta\lambda_C = \frac{\lambda_0}{N \cdot n}}$

System Design

In the case where spectral purity (slit width) limits the resolving power: $\Delta\lambda_s = -\cos\alpha \frac{W'}{f_{coll}} \frac{d}{n}$

The resolving power is given by: $R_s \equiv \frac{\lambda}{\Delta\lambda_s} = \frac{n\lambda}{d} \frac{f_{coll}}{W'} \frac{1}{\cos\alpha}$

If we operate in Littrow mode, then $\alpha \sim \beta$ and the grating equation becomes: $\frac{n\lambda}{d} = 2\sin\alpha$

So, the resolving power for the slit limited case is then:

$$R_s = 2\tan\alpha \frac{f_{coll}}{W'} = 2\tan\alpha \frac{f^{\#}_{col}}{W'} G_{proj}$$

Projected grating size

System Design

But, we have:

$$\frac{f^{\#}_{coll}}{W'} = \frac{f^{\#}_{tel}}{W'_{tel}} = \frac{f^{\#}\text{of telescope}}{\text{size of slit in telescope focal plane}}$$

$$= \frac{1}{\theta_{beam} D_{tel}}$$

Putting this together we have:

$$\boxed{R_s = \frac{2\tan\alpha \cdot G_{proj}}{\theta_{beam} D_{tel}}}$$

Maximum Resolving Power

What is the maximum resolving power attainable with a diffraction grating? The chromatic resolving power is:

$$R_c \equiv \frac{\lambda}{\Delta\lambda_c} = \frac{Gn}{d} = nN$$

The maximum resolving power is achieved by setting the entrance and exit slit widths equal to a diffraction limited spot on the sky:

$$\lambda \approx \theta_{beam} D_{telescope}$$

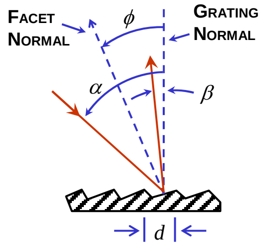
$$\Rightarrow \theta_{beam} = \frac{\lambda}{D_{tel}}$$

$$W = \theta_{beam} \cdot (\text{focal length}) = \theta_{beam} \cdot D_{tel} \cdot \text{f/\#}$$

$$\Rightarrow \text{W} = \text{f/\#} \cdot \lambda$$

So given the f/# in the system, the maximum resolving power is attained by setting ***W* = *f/#*•*λ***

Grating Equation Summary



- d = groove spacing
- ϕ = blaze angle
- f_{cam} = camera focal length
- f_{coll} = collimator focal length
- β_B = angle at which blaze is directed
- λ_o = blaze λ for $\alpha = \beta$ (Littrow)
- λ_B = wavelength of true blaze
- w = slit image width
- W′ = entrance slit width
- G = grating width
- G_{proj} = projected grating width

GRATING EQUATION

$$\frac{n\lambda}{d} = \sin\alpha + \sin\beta$$

ANGULAR DISPERSION

$$\frac{d\beta}{d\lambda} = \frac{n}{d\cos\beta}$$

LINEAR DISPERSION

$$\frac{d\lambda}{dx} = \frac{1}{f_{cam}}\left[\frac{d\beta}{d\lambda}\right]^{-1} = \frac{1}{f_{cam}}\frac{d\cos\beta}{n}$$

1st ORDER LITTROW

$$\frac{\lambda_o}{d} = 2\sin\phi \qquad (\alpha = \beta)$$

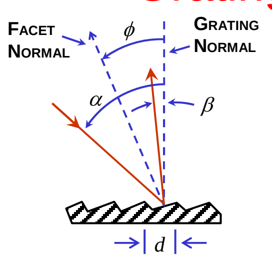
WAVELENGTH OF TRUE BLAZE

$$\frac{\lambda_B}{\lambda_o} = \cos\tfrac{1}{2}(\alpha_B - \beta_B) \qquad (\alpha_B + \beta_B = 2\phi)$$

SLIT MAGNIFICATION

$$\frac{w}{W'} = -\frac{\cos\alpha}{\cos\beta}\frac{f_{cam}}{f_{coll}}$$

Grating Equation Summary



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- w = slit image width
- W′ = entrance slit width
- G = grating width
- G_{proj} = projected grating width

CHROMATIC RESOLUTION

$$\Delta\lambda_c = \frac{\lambda}{G}\frac{d}{n}$$

$$R_c \equiv \frac{\lambda}{\Delta\lambda_c} = \frac{2G_{proj}\tan\alpha}{\lambda}$$

SPECTRAL PURITY

$$\Delta\lambda_s = -\cos\alpha\frac{W'}{f_{coll}}\frac{d}{n}$$

$$R_S \equiv \frac{\lambda}{\Delta\lambda_c} = \frac{2G_{proj}\tan\alpha}{\theta_{beam}D_{tel}}$$

- D_{tel} = telescope diameter
- θ_{beam} = angular size of slit on the sky

$$R_c = \frac{\lambda}{\Delta\lambda_c}$$

$$= \frac{Gn}{d}$$

$$= \frac{2 \tan\alpha \; G_{proj}}{\lambda}$$

by the grating eq'n (in Littrow) and using $G_{proj} = G\cos\alpha$

The is a critical wavelength, $\lambda \equiv \lambda_c$, at which $R_s = R_c$

$$\lambda = \theta_B D_{TEL}$$

$$= 4.85\mu m \;\; \theta_B('') \; D_{TEL}(m)$$

In a nutshell, the maximum resolving power is attained when you use a diffraction limited slit

For $\lambda < \lambda_c$, $R_s < R_c$

$\lambda > \lambda_c$, $R_s > R_c$

The maximum resolution of the system will be set by R_s or R_c, whichever is smaller.