Prisms

Astronomy 525

Lecture 10

Outline

- Prisms
- Basic Principles
- The prism spectrograph
- Examples

- Reference: Kitchin, C.R. “Astrophysical Techniques” 2nd Edition. Adam Hilger, Bristol, Philadelphia and New York 1991. Chapter 4.

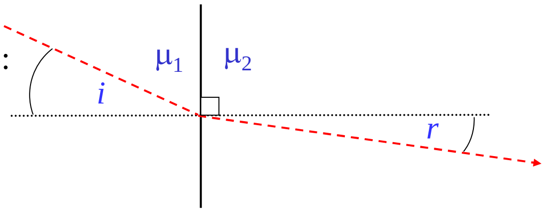
Snell's Law

- Prisms are unique from gratings, FPI, or FTS -- they employ differential refraction rather than interference to achieve spectral resolution.
- Recall Snell's law, that relates the angles of incidence, *i*, and refraction, *r*, through the refractive indices in the two media, μ_1, and μ_2 :

$$\mu_1 \sin i = \mu_2 \sin r$$

- If the first medium is air, this reduces to:

$$\frac{\sin i}{\sin r} = \mu_2$$



Glass Dispersion

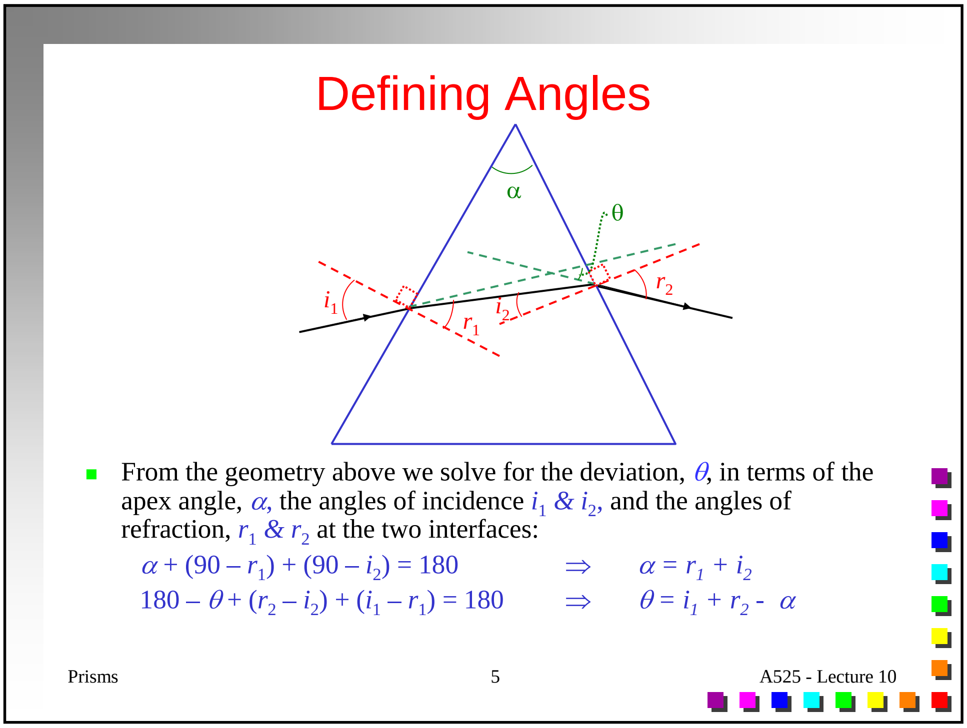
- Over a restricted wavelength interval, the index of refraction of many glasses can be approximated by the Hartmann dispersion formula:

$$\mu_\lambda = A + B/(\lambda - C)$$

- where the Hartmann constants A, B, and C for crown and flint glass in the optical are:

Glass Type	A	B (μm)	C (μm)
Crown	1.477	0.0320	-0.0210
Dense Flint	1.603	0.0208	0.0143

- Therefore, white light is dispersed at the interface with longer wavelengths refracted less than the shorter ones.
- For prisms we want "chromatic aberration", a change in the deviation of incident rays with wavelength.

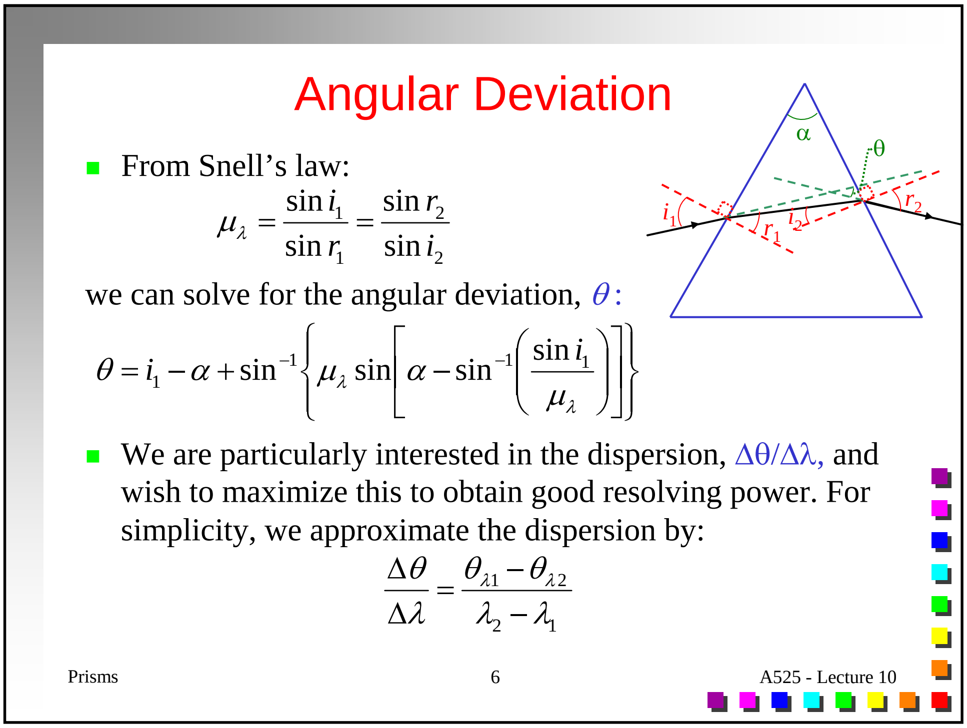


Defining Angles

- From the geometry above we solve for the deviation, θ, in terms of the apex angle, α, the angles of incidence i_1 & i_2, and the angles of refraction, r_1 & r_2 at the two interfaces:

$$\alpha + (90 - r_1) + (90 - i_2) = 180 \quad \Rightarrow \quad \alpha = r_1 + i_2$$

$$180 - \theta + (r_2 - i_2) + (i_1 - r_1) = 180 \quad \Rightarrow \quad \theta = i_1 + r_2 - \alpha$$



Angular Deviation

- From Snell's law:

$$\mu_\lambda = \frac{\sin i_1}{\sin r_1} = \frac{\sin r_2}{\sin i_2}$$

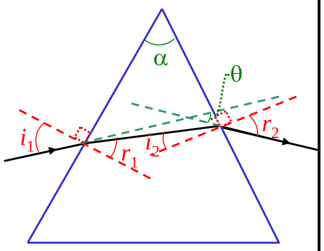
we can solve for the angular deviation, θ:

$$\theta = i_1 - \alpha + \sin^{-1}\left\{\mu_\lambda \sin\left[\alpha - \sin^{-1}\left(\frac{\sin i_1}{\mu_\lambda}\right)\right]\right\}$$

- We are particularly interested in the dispersion, $\Delta\theta/\Delta\lambda$, and wish to maximize this to obtain good resolving power. For simplicity, we approximate the dispersion by:

$$\frac{\Delta\theta}{\Delta\lambda} = \frac{\theta_{\lambda 1} - \theta_{\lambda 2}}{\lambda_2 - \lambda_1}$$

Dispersion I



- For the case of a dense flint prism

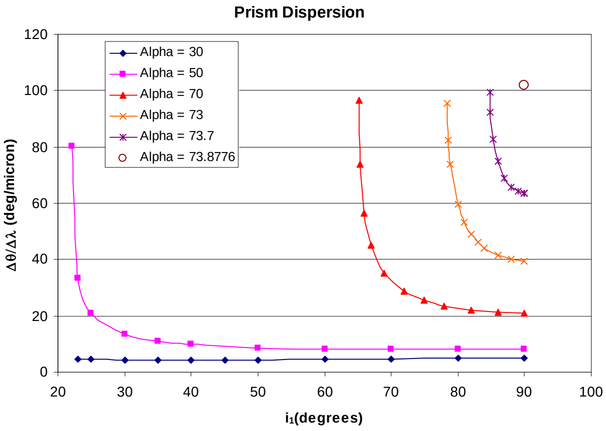
λ (μm)	μ
0.486	1.664
0.589	1.650

- So that the dispersion is (deg/μm)

$$\frac{\Delta\theta}{\Delta\lambda} = 9.71\cdot\left[\sin^{-1}\left\{1.664\sin\left[\alpha-\sin^{-1}\left(\frac{\sin i_1}{1.664}\right)\right]\right\} - \sin^{-1}\left\{1.650\sin\left[\alpha-\sin^{-1}\left(\frac{\sin i_1}{1.650}\right)\right]\right\}\right]$$

Dispersion II

Right: Variation of the dispersion, Δθ/Δλ with angle of incidence for a dense flint prism, for various apex angles, α.



- The dispersion is a function of the apex angle, α, and the angle of incidence, i_1, with the maximum dispersion for dense flint of 102 deg/μm occurring at an incidence angle of 90° and an apex angle of 73.8776°
- This is grazing incidence, and exitance from the prism, and one can show that the ray passes through the prism symmetrically.

Operating Range

- The angle of incidence for the maximum dispersion case is ~ 90°. This ***is not desired*** since the reflection losses will be prohibitive. (As we will see later, the resolving power will also approach zero).
- The symmetrical passage of the rays through the prism ***is desired***, since the dispersion is maximized, and the astigmatism is minimized in this configuration. The condition of symmetric passage for any prism is typically referred to as the position of minimum deviation, as show on the next viewgraphs.

Minimum Deviation

- Begin by rewriting θ in terms of r_1 instead of i_1:

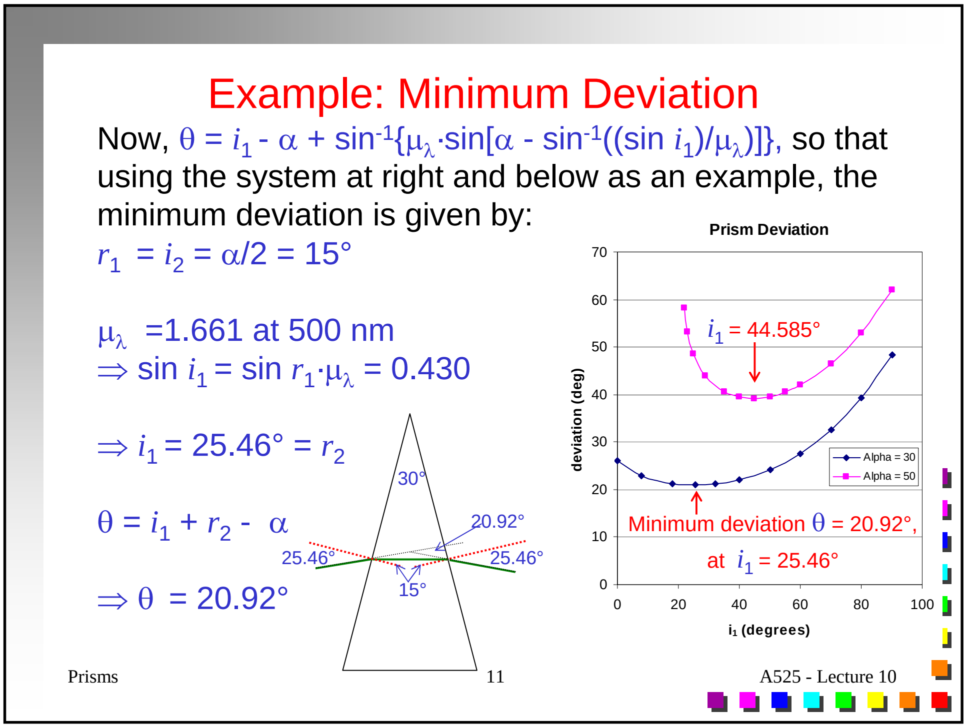
$$\theta = \sin^{-1}\left[\mu_\lambda \sin r_1\right] + \sin^{-1}\left[\mu_\lambda \sin\left(\alpha - r_1\right)\right] - \alpha$$

- For an extremum: $\partial\theta / \partial r_1 = 0$

$$\Rightarrow \frac{\mu_\lambda \cos r_1}{\sqrt{1-\left(\mu_\lambda \sin r_1\right)^2}} - \frac{\mu_\lambda \cos\left(\alpha - r_1\right)}{\sqrt{1-\left(\mu_\lambda \sin\left(\alpha - r_1\right)\right)^2}} = 0$$

$$\Rightarrow \left[\cos 2r_1 - \cos 2\left(\alpha - r_1\right)\right]\left(1 - 2\mu_\lambda^2\right) = 0$$

- Which has $r_1 = \alpha/2$ as a solution. But $\alpha = r_1 + i_2$, we have $i_2 = \alpha/2 \Rightarrow r_1 = i_2$.
- Therefore, the minimum deviation occurs for symmetrical passage of the ray through the prism.



Example: Minimum Deviation

Now, $\theta = i_1 - \alpha + \sin^{-1}\{\mu_\lambda \cdot \sin[\alpha - \sin^{-1}((\sin i_1)/\mu_\lambda)]\}$, so that using the system at right and below as an example, the minimum deviation is given by:

$r_1 = i_2 = \alpha/2 = 15^\circ$

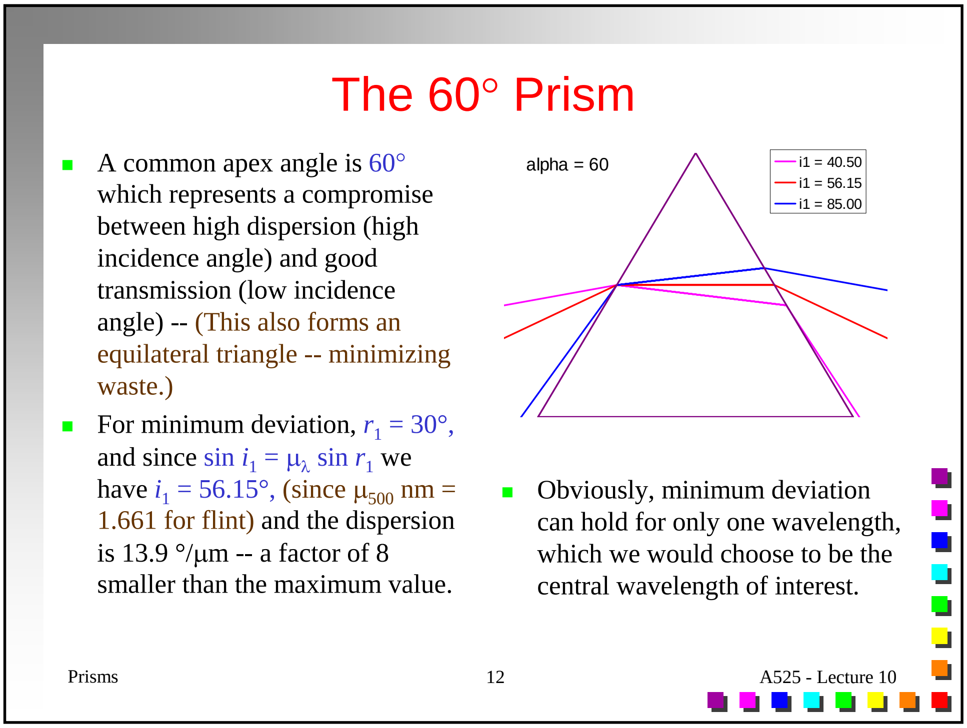
$\mu_\lambda = 1.661$ at 500 nm

$\Rightarrow \sin i_1 = \sin r_1 \cdot \mu_\lambda = 0.430$

$\Rightarrow i_1 = 25.46^\circ = r_2$

$\theta = i_1 + r_2 - \alpha$

$\Rightarrow \theta = 20.92^\circ$



The 60° Prism

- A common apex angle is 60° which represents a compromise between high dispersion (high incidence angle) and good transmission (low incidence angle) -- (This also forms an equilateral triangle -- minimizing waste.)
- For minimum deviation, $r_1 = 30^\circ$, and since $\sin i_1 = \mu_\lambda \sin r_1$ we have $i_1 = 56.15^\circ$, (since μ_{500} nm = 1.661 for flint) and the dispersion is 13.9 °/µm -- a factor of 8 smaller than the maximum value.

- Obviously, minimum deviation can hold for only one wavelength, which we would choose to be the central wavelength of interest.

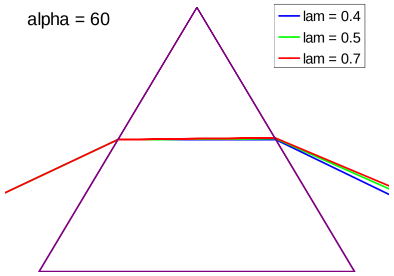
Deviation Angle vs. Wavelength

- How does the deviation angle vary with wavelength?
- Consider a prism with apex angle of 60°. At minimum deviation, we then have: $r_1 = 30°$. The deviation is then ($r_1 = \alpha/2$):

  $\theta = \sin^{-1}\{\mu_\lambda \cdot \sin r_1\} + \sin^{-1}\{\mu_\lambda \sin(\alpha - r_1)\} - \alpha$

  $= 2\sin^{-1}(\mu_\lambda \sin r_1) - \alpha$
- With $r_1 = 30°$, and $\alpha = 60°$, and the Hartmann formula for μ_λ:

  $\theta = 2\sin^{-1}\{[A + B/(\lambda - C)]/2\} - 60°$



Deviation vs. λ (continued)

$$\theta = 2\sin^{-1}\{\tfrac{1}{2}[A + B/(\lambda - C)]\} - 60°$$

We can approximate the change in angle w.r.t λ by:

$$d\theta/d\lambda = -180B/\{\pi(\lambda - C)^2 \underbrace{[1 - \tfrac{1}{4}[A + B/(\lambda - C)]^2]^{1/2}}\} \quad [°/m]$$

Since $A + B/(\lambda - C) = \mu_\lambda \sim 1.5 \Rightarrow \overbrace{A + B/(\lambda - C)}$

30 × less than 1st term

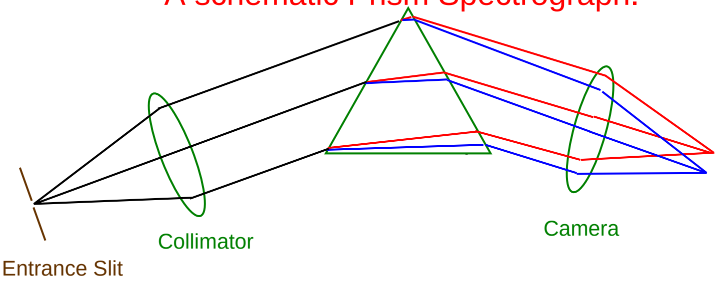
$$\Rightarrow d\theta/d\lambda = -180AB/\{\pi(\lambda - C)^2\} - \overbrace{180B^2/(\lambda - C)^3}$$

$$\Rightarrow d\theta/d\lambda = -180AB/\{\pi(\lambda - C)^2\}, \text{ or } \boxed{d\theta/d\lambda \propto 1/(\lambda - C)^2}$$

Basic Prism Spectrograph

The dispersion therefore rapidly increases towards shorter wavelengths -- it is almost 5 times larger at 400 nm than at 700 nm for a dense flint prism.

A schematic Prism Spectrograph.



Note: Lenses should be achromatic, or else chromatic aberration will cause the focal plane to be tilted.

Dispersion in the Focal Plane

The linear dispersion is the distance moved in the focal plane per unit wavelength interval.

$$dx/d\lambda = f_2 \cdot d\theta/d\lambda \quad (\theta \text{ small, measured in radians})$$

$$= -\,180AB\, f_2 / \{\pi(\lambda - C)^2\}$$

Typically, the reciprocal linear dispersion $d\lambda/dx$, is quoted, and it will have a value in the range:

$$10^{-7} < d\lambda/dx < 5 \times 10^{-5} \qquad \text{(unit-less)}$$

or, the common unit is 0.1 to 50 nm/mm

Spectral Purity (Slit Limited Resolution)

The resolving power will be limited by the resolving power of the optics, and the projected slit width. If we let:

s = slit width
f_1 = collimator focal length
f_2 = cameral focal length

Then the projected slit width will be: $W = s(f_2/f_1)$

And the spectral purity is: $W(d\lambda/dx) = W\,(d\lambda/d\theta)/f_2$.

Rayleigh Resolution Limit

If L is the length of the prism face, then the width of the light intercepted by the camera is:

$$\begin{aligned} D &= L[\cos r_2] \\ &= L[1-\sin^2 r_2]^{1/2} \\ &= L[1-\mu_\lambda^2 \sin^2(\tfrac{1}{2}\alpha)]^{1/2} \quad \text{(since } r_1 = i_2 = \alpha/2) \end{aligned}$$

The diffraction of the light will prevent perfect imaging, so that the linear Rayleigh limit of resolution, W_R is:

$$W_R = f_2\,\lambda/D = f_2\,\lambda/\{L[1-\mu_\lambda^2 \sin^2(\tfrac{1}{2}\alpha)]^{1/2}\}$$

If D is limited by some other element in the optical system, then the Raleigh criterion is evaluated for that aperture.

Maximum Resolving Power

For the highest resolving power, we chose:

$$W_R = W \Rightarrow f_2\,\lambda/D = s\,(f_2/f_1)$$
$$\Rightarrow s = f_1\,\lambda/D = f_1\,\lambda/\{L[1-\mu_\lambda^2 \sin^2(\tfrac{1}{2}\alpha)]^{1/2}\}$$

For this case, the spectral resolution, denoted by W_λ, is:

$$W_\lambda = W_R\, d\lambda/dx$$
$$= (f_2\,\lambda/D)\cdot\{1/f_2 \cdot d\lambda/d\theta\} = \lambda/D \cdot d\lambda/d\theta$$
$$\sim \lambda\,(\lambda - C)^2/\{ABL\{1-[A + B/(\lambda-C)]^2\sin^2(\alpha/2)\}^{1/2}\}$$

The resolving power is given by:

$$R = \lambda/W_\lambda \sim \{ABL\{1-[A + B/(\lambda - C)]^2\sin^2(\alpha/2)\}^{1/2}\}/(\lambda - C)^2$$

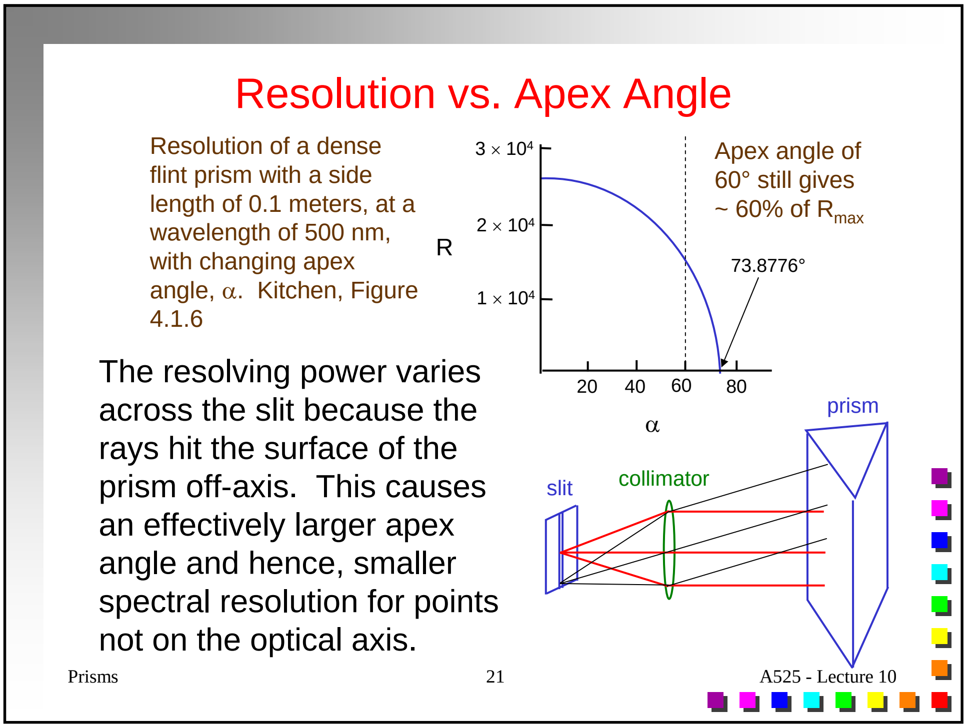
Notice that this is linear in L. The bigger the prism, the higher resolving power you can achieve.

Example: 60° Prism Size

For a dense flint prism, with an apex angle of 60°, and a side length of 0.1 meters, in the visible, R ~ 15,000

Notice that in the expression for R, the $L\{\ldots\}^{1/2}$ term is just the width of the beam going through the camera (= D). For maximum dispersion, $i_1 \sim r_2 \sim 90°$, so that this term goes to zero (i.e. the emergent beam width goes to zero, $D = L\cos(r_2) \Rightarrow R \to 0$).

60° apex angle is ~ optimal. As mentioned before, there is a trade-off between resolution, dispersion, and throughput.

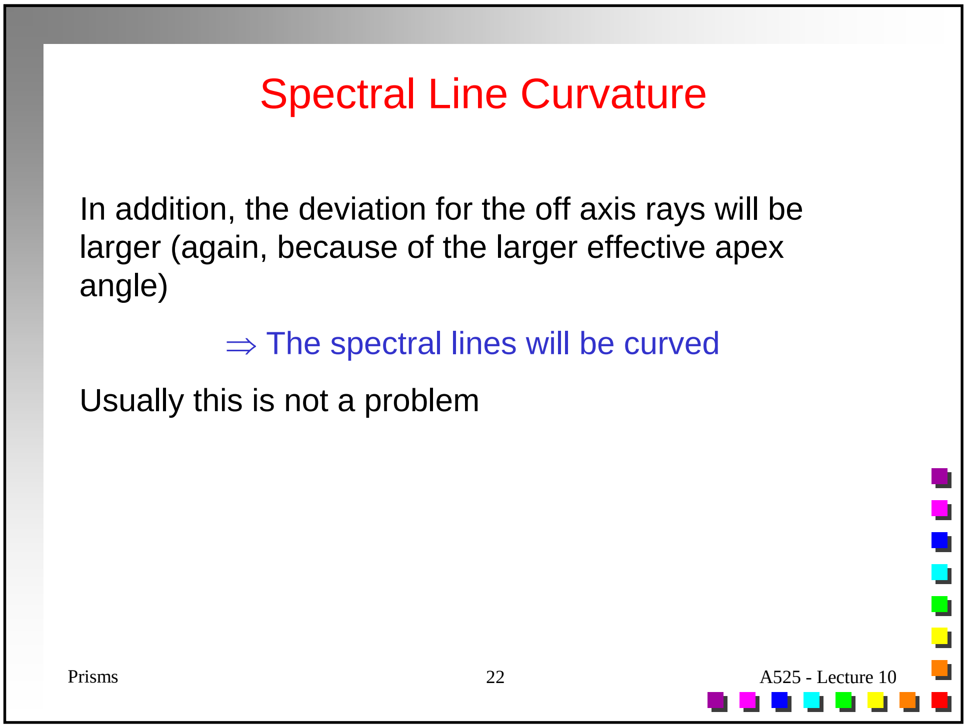


Resolution vs. Apex Angle

Resolution of a dense flint prism with a side length of 0.1 meters, at a wavelength of 500 nm, with changing apex angle, α. Kitchen, Figure 4.1.6

Apex angle of 60° still gives ~ 60% of R_{max}

The resolving power varies across the slit because the rays hit the surface of the prism off-axis. This causes an effectively larger apex angle and hence, smaller spectral resolution for points not on the optical axis.



Spectral Line Curvature

In addition, the deviation for the off axis rays will be larger (again, because of the larger effective apex angle)

⇒ The spectral lines will be curved

Usually this is not a problem

Pluses and Minuses of Prisms

Why Prisms?

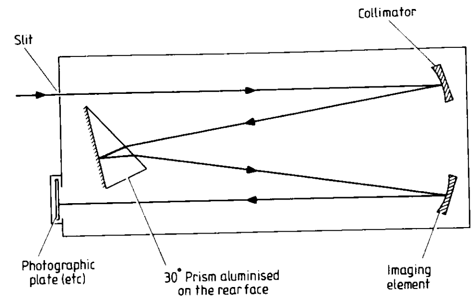
No order sorting is required as with a grating or Fabry-Perot. Therefore, one can do more than one octave in wavelength at the same time.

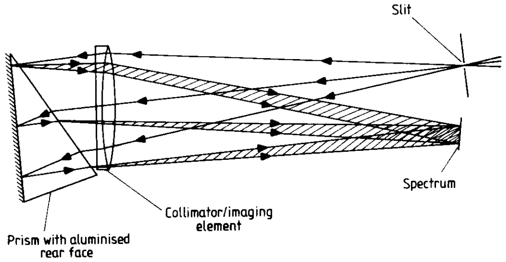
Problems:

1. Materials selection is good in the optical (glass) and near-IR (crystal quartz), but poor in the mid and far-IR. Also, there are severe problems with absorption in the UV (fused silica only goes to 200 nm).

2. In general, prisms have less throughput than gratings or Fabry-Perot interferometers.

Double Pass & Littrow Spectrographs



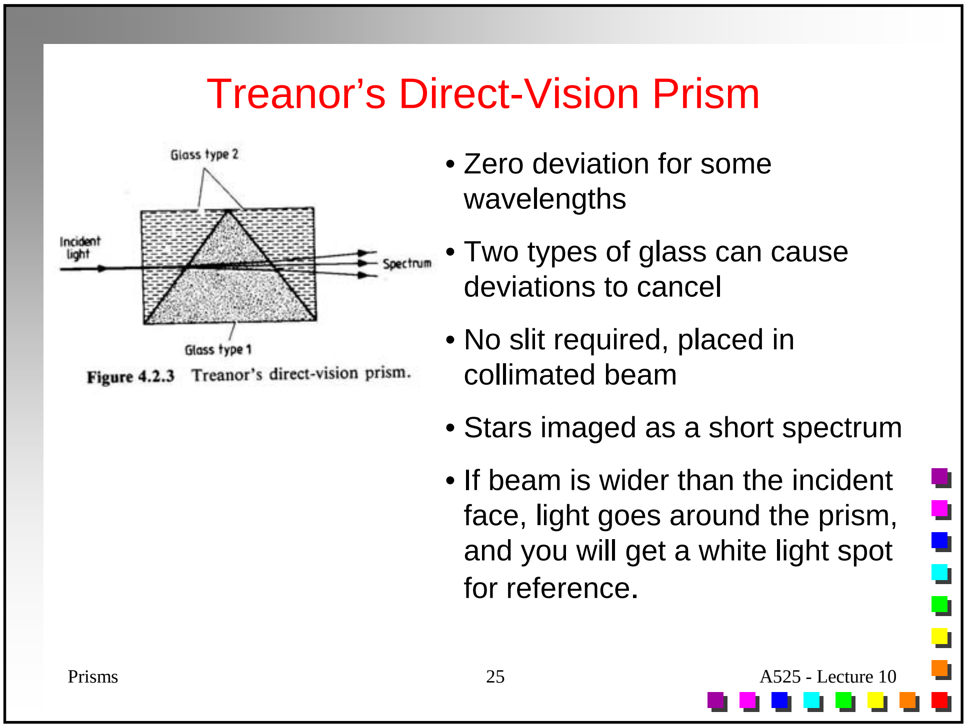


Double Pass Prism Spectrograph

More compact than a simple prism spectrograph. Same dispersion as 60° prism, smaller optic, but minimum deviation doesn't exist so that there will be astigmatism

Littrow Spectroscope

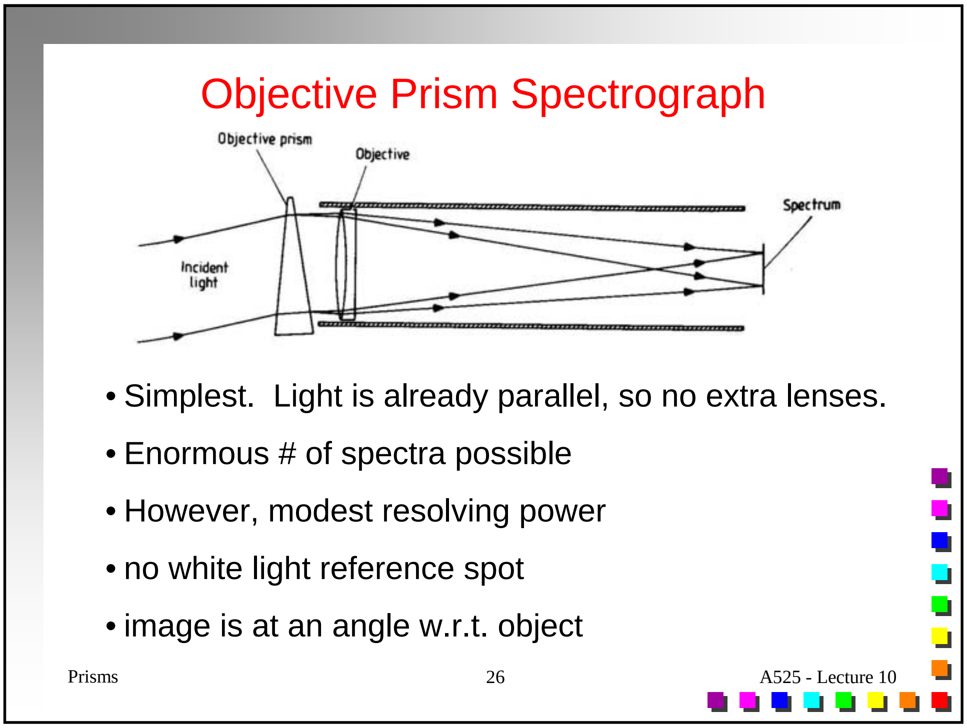
Single lens or mirror minimizing cost and size. Baffling can be challenging.



Treanor's Direct-Vision Prism

Figure 4.2.3 Treanor's direct-vision prism.

- Zero deviation for some wavelengths
- Two types of glass can cause deviations to cancel
- No slit required, placed in collimated beam
- Stars imaged as a short spectrum
- If beam is wider than the incident face, light goes around the prism, and you will get a white light spot for reference.



Objective Prism Spectrograph

- Simplest. Light is already parallel, so no extra lenses.
- Enormous # of spectra possible
- However, modest resolving power
- no white light reference spot
- image is at an angle w.r.t. object