Fabry-Perot Interferometers: I

Astronomy 525

Lecture 13

Outline

Fabry-Perot Interferometers: I

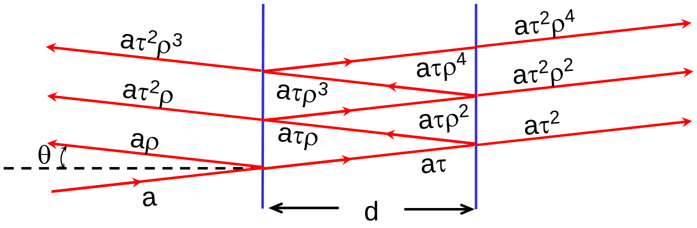
- Theory
- Limits to Performance

- Part II will include several examples

- Literature:
  - C.R. Kitchin, “Astrophysical Techniques”
  - (Born & Wolf, “Principles of Optics”)

Theory: 1

Fabry-Perots are best thought of as resonant cavities formed between two flat, parallel highly reflecting mirrors. The pair of mirrors is called an *etalon.*



Let: d = spacing between the mirrors
τ = amplitude transmissivity
ρ = amplitude reflectivity

For a collimated beam with amplitude, a, incident at angle θ, the phase shift (difference) between successive transmitted rays is:

$$\delta = \frac{2\pi}{\lambda} 2d \cdot \cos\theta$$

Theory: 2

Here we ignored the phase shift on reflection (this just acts like a change in the cavity size) and assumed that the index of refraction between the mirrors is 1.

The reflection phase shift can be included by letting:

$$d \to d + \Delta d$$

where:

$$\Delta d = -\frac{\phi \cdot \lambda}{2\pi} \cos\theta$$

and ϕ is the reflection phase shift (excepting the first ray).
The complex amplitude for the nth transmitted ray is:

$$\psi_n = a\tau^2\rho^{2(n-1)}e^{i(\omega t-(n-1)\delta)}$$

and the total amplitude is:

$$\psi = \sum_{n=1}^{\infty} \psi_n = a\tau^2 e^{i\omega t}/(1 - \rho^2 e^{-i\delta})$$

Theory: 3 (Airy Function)

So that the transmitted intensity is:

$$I \equiv \psi^*\psi = a^2\tau^4/(1 - 2\rho^2\cos\delta + \rho^4)$$

Rewriting in terms of the mirror intensity coefficients:

$$t \equiv \tau^2 \quad \text{and} \quad r \equiv \rho^2$$

The transmitted intensity of the etalon is:

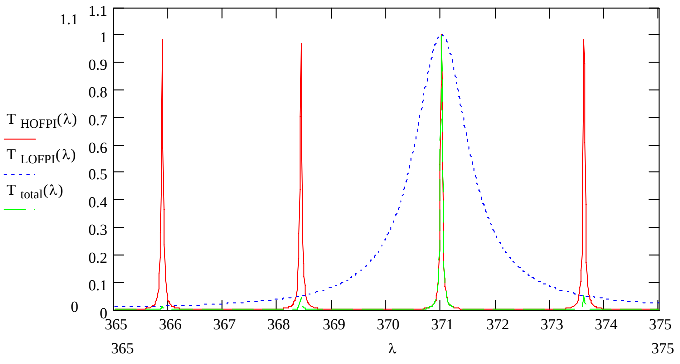
$$I(\delta)/I_o = T_{max}/\{1 + [4r/(1 - r)^2]\cdot\sin^2(\delta/2)\}$$

where: $T_{max} \equiv t^2/(1 - r)^2$

The form above is that of the **Airy function**. The incident intensity is given by:

$$I_o = a^2$$

Theory: 3 (Airy Function Transmission)



Theory: 4 (Peak Width)

In the limit of no absorption:

$$t = 1 - r \text{ and } T_{max} = 1$$

The Airy function has peak when the phase shift, δ, is an integer multiple of 2π, e.g.

$\delta = 2\pi m$ m = 0, 1, 2, … = order of etalon

where: $m = 2d/\lambda$

for normal incidence.

The full-width at half-maximum (FWHM) of a single peak is:

$$\delta_{FWHM} = 4 \sin^{-1}((1 - r)/2\sqrt{r}) \approx 2(1 - r)/\sqrt{r}$$

Theory: 5 (Reflectivity Finesse)

This approximation is accurate to within 1% for r > 0.65. For normal incidence, we can compute the single peak width, from the definition of δ:

$$\begin{aligned}\Delta\delta &= (4\pi d\cdot\Delta\lambda_{FWHM})/\lambda^2 \\ &\equiv \delta_{FWHM}\end{aligned}$$

so that: $$\Delta\lambda_{FWHM} \approx \frac{\lambda^2}{2d}\cdot\frac{1-r}{\pi\sqrt{r}}$$

The finesse, F, of an FPI is the ratio of the distance between peaks, to their FWHM

$$\boxed{\begin{aligned}F &\equiv 2\pi/\delta_{FWHM} \approx \pi\sqrt{r}/(1 - r) \\ &= \lambda^2/(2d\cdot\Delta\lambda_{FWHM}) = \lambda/(m\cdot\Delta\lambda_{FWHM})\end{aligned}}$$

The finesse along with the transmission, is the primary figure of merit for an FPI.

Theory: 6 (Free Spectral Range)

The resolving power:

$$R_{FPI} \equiv \lambda/\Delta\lambda_{FWHM}$$

$$= m \cdot (\lambda/(m\Delta\lambda_{FWHM})) = m \cdot F$$

F can be thought of as a path length multiplier, and is roughly the number of reflections an average photon makes before being transmitted through the system.

The spacing between transmission peaks (in units of wavelengths) is called the **free spectral range**. We have:

$$m\lambda = (m - 1) \cdot (\lambda + \Delta\lambda_{FSR})$$

$$\Rightarrow \Delta\lambda_{FSR} = \lambda/m$$

Thus: $FSR = \lambda/m = \lambda^2/(2d)$

Theory: 7

Therefore, both the bandwidth (1/(resolving power)) and the distance between peaks (FSR) scale inversely with plate separation d. Their ratio is a constant. In other words, when the resolution gets very large, the orders of an FPI will tend to overlap.

Theory: 8 (Contrast)

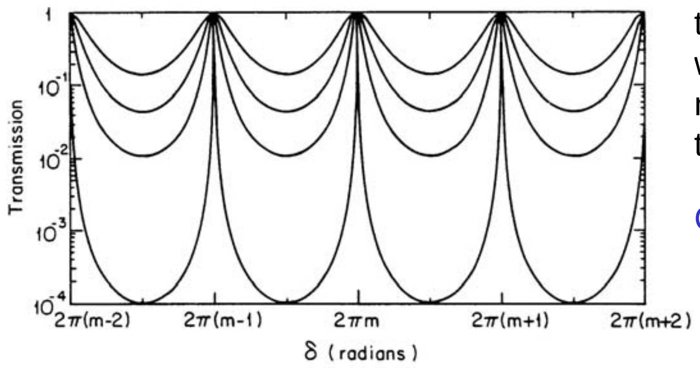


Figure A–2. The log of the etalon transmission for intensity reflectivities of 0.45, 0.65, 0.81 and 0.98 versus the phase shift between successive paths. The maximum to minimum transmission ratios, or contrasts, are 6, 20, 100, and 10,000 respectively.

Graf 1986 Phd Thesis, Cornell

Another figure of merit for the FPI is the contrast, which is the ratio of the maximum to minimum transmission:

$$C \equiv T_{max}/T_{min}$$

$$= (1 + r)^2/(1 - r)^2$$

$$= 1 + (2F/\pi)^2$$

For a $F > 15$, $C > 100$, which is adequate.

C is a strong function of F: $F > 45 \Rightarrow C > 1000$

Limits to Performance: I

1. Mirror Absorption

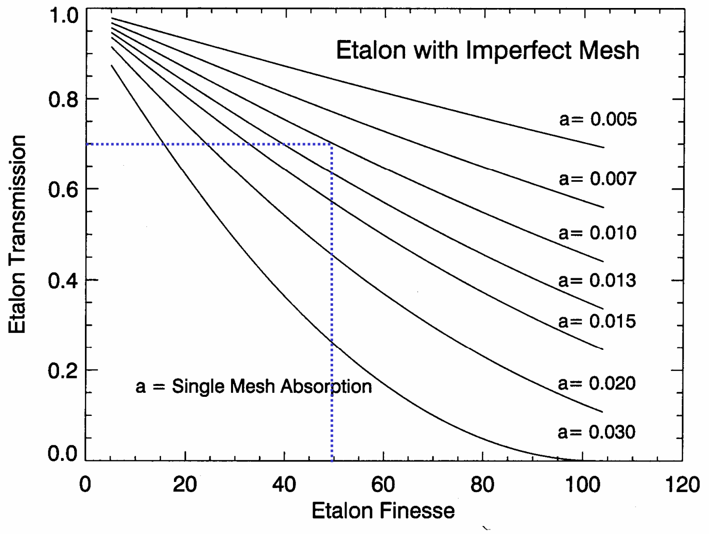
If a is the single pass intensity absorption coefficient, then the maximum transmitted intensity is:

$$T_{max} = \{t/(t + a)\}^2$$
$$= \{(1 - r - a)/(1 - r)\}^2$$

The reflectivity finesse is independent of the absorption coefficient. Note that for small a ($\leq 2\%$), the transmission is $1 - aF$, since F ~ the number of reflections. For $aF/(\pi\sqrt{r}) << 1$:

$$T_{max} \sim 1 - 2aF/(\pi\sqrt{r})$$

Limits to Performance: II



Bradford 2001 Phd Thesis, Cornell

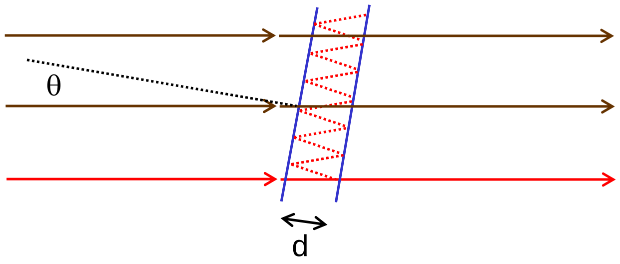
Figure 3.5: Transmission of an FPI etalon as a function of the single-pass absorption coefficient $a = 1 - t - r$ of a single mesh. For the Buckbee-Mears meshes we use, $a \sim 0.01$, and the lower finesse cavities result in substantially higher transmission. For this reason, a triple FPI system such as SPIFI results in a better transmission than a double of the same resolving power.

Mirror Absorption

Transmission of an FPI as a function of $a = 1 - t - r$. Typically $a \sim 1\%$, for which a good FPI will have F ~ 30 to 50 ($r = 0.90 \rightarrow 0.94$)

It is important to keep T_{max} close to unity, since large F is not useful.

Limits to Performance: III



2. Etalon Tip

Tipping the etalon causes beams to “walk” off the mirrors. When the central ray has traveled a distance equal to the beam radius across the etalon, the constructive interference is significantly reduced:

$$D/2 \approx F_{Tip} \cdot d \cdot \sin(\theta_{Tip}) \approx F_{Tip} \cdot d\ \theta_{Tip}$$

$$\Rightarrow F_{Tip} \approx D/(2d \cdot \theta_{Tip})$$

where D is the beam diameter.

Limits to Performance: IV

Etalon Tip

Notice that tipping the etalon shifts the central wavelength of transmission to shorter wavelength:

$\theta_{Tip} = 0$: $2d/\lambda_o = m$

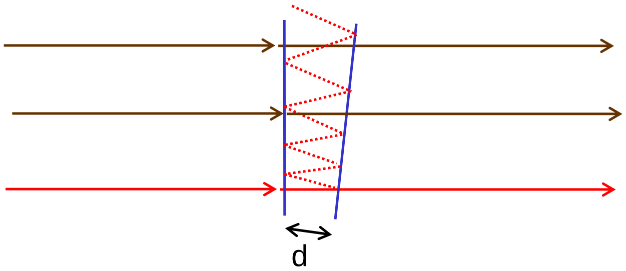
θ_{Tip} small: $2d/\lambda_1(\cos\theta) = m$

$\Rightarrow 2d/\lambda_1(1 - \theta^2/2) = m$

$\Rightarrow \lambda_1 - \lambda_o \equiv \Delta\lambda = -\lambda_o\theta^2_{Tip}/2$

$\Rightarrow \Delta\lambda = -\lambda_o\theta^2_{Tip}/2$ **(blue shift)**

Limits to Performance: V



3. Mirror Parallelism

If one mirror is tipped with respect to the other at an angle θ, then the problem is much more severe than a mere tipping of the entire etalon. Not only does the reflected angel grow with each reflection, but the cavity resonates at a different wavelength at different positions!!

Limits to Performance: VI

Mirror Parallelism

To make an estimate of this effect, lets assume resonance at λ_o at one end of the etalon, and at λ_1 at the other. The maximum resolution of the etalon is then:

$$R_{max} = (\lambda_o + \lambda_1)/\{2(\lambda_o - \lambda_1)\} = F_{Par} \cdot m$$

Now, if Δ is the gap difference, i.e. $\Delta = D \sin\theta$, then:

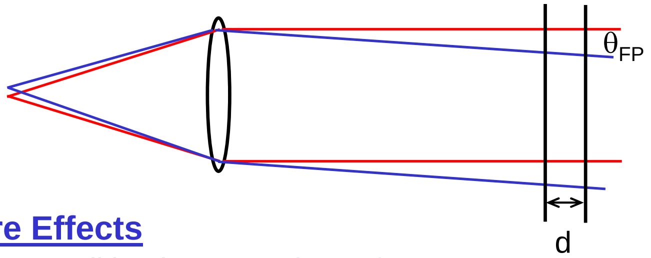
$$(\lambda_o - \lambda_1) = 2d/m - 2(d + \Delta)/m = -2\Delta/m$$

or, letting $(\lambda_o + \lambda_1)/2 \rightarrow \lambda$, then:

$$F_{par} \cdot m = m\lambda/(2\Delta) \Rightarrow \boxed{F_{Par} = \lambda/(2\Delta)}$$

If the mirrors are parallel to λ/n then $F_{Par} = n/2$
$\Rightarrow$ you need parallelism to better than $\lambda/200$ or so!

Limits to Performance: VII



4. Aperture Effects

The resonant condition is: $2d\cdot\cos\theta/\lambda = m$
normal ray: $2d/\lambda_o = m$
off-axis ray: $2d\cdot\cos\theta_{FP}/(\lambda_1) = m$

$\Rightarrow 2d/\lambda_o \approx 2d/\lambda_1 (1 - \theta_{FP}^2/2) = m$

$\Rightarrow \lambda_1 - \lambda_o \equiv -\Delta\lambda = - \lambda_o\theta_{FP}^2/2$

$\Rightarrow \Delta\lambda/\lambda_o \equiv 1/R = \theta_{FP}^2/2 \equiv 1/(F_{aperture}\cdot m)$

$\Rightarrow F_{aperture} = R/m = 2\pi/\{m\pi(\theta_{FP}^2)\} = \pi\lambda_o/(d\cdot\Omega_{FP})$

$\Rightarrow R = 2\pi/\Omega_{FP}$, or $R\cdot\Omega_{FP} = 2\pi$

Limits to Performance: VIII

Aperture Effects

Now, since etendue is conserved:

$$A_{FP}\Omega_{FP} = A_{Primary}\Omega_{Primary}$$

$$\Rightarrow \Omega_{Primary} \cdot A_{Primary}/A_{FP} \cdot R = 2\pi$$

$$\Rightarrow R_{max} = 2\pi\{A_{FP}/A_{Primary}\}/\Omega_{Primary}$$

5. Diffraction Effects

Diffraction results in a minimum angle within the beam given by: $\theta_{Teldiff} \sim \lambda/D_{Tel}$. Plugging this into our aperture effects equation:

$$R_{diff} = 2\pi\{A_{FP}/A_{Primary}\}/\{\pi/4 \cdot (\lambda/D_{Primary})^2\} = 8 \cdot (D_{FP}/\lambda)^2$$

$$F_{diff} = R_{diff}/m = \lambda/(2d) \cdot 8 \cdot (D_{FP}/\lambda)^2 = 4 \cdot D_{FP}^2/(d \cdot \lambda)$$

where: $\Omega = \pi/4 \cdot (\lambda/D)^2$, D_{FP} is the diameter of the beam through the etalon, and d is the plate spacing.

Limits to Performance: IX

6. Surface Defects

Irregularities in the surfaces of the mirrors will cause irregularities in the path lengths for different regions of the cavity. Suppose we have an rms gap variation that is given by Δs, and is distributed in a Gaussian manner. It can be shown that:

$$F_{surf} = \lambda/\{(32(\ln 2))^{1/2}\Delta s\} \approx \lambda/(4.7\Delta s)$$

Limits to Performance: X

Effective Finesse and Transmission

Each effect reduces both the FPI transmission, and the finesse. The total effective finesse is given by the inverse quadratic sum of each finesse term above with the reflective finesse (Cooper, J. and Greig, J.R. 1963 *J. Sci. Instrum.,* 40, 433.):

$$1/F_{\mathrm{eff}}^2 = \Sigma\, 1/F_i^2$$

The peak transmission is reduced, since some of the radiation is absorbed, some of it is forced out of the beam, and some of it interferes destructively. The etalon transmission is reduced by the ratio of the effective finesse to the reflective finesse:

$$\begin{aligned} T_{\mathrm{FPI}} &= T_{\max}\cdot\{F_{\mathrm{eff}}/F_{\mathrm{ref}}\} \\ &= [(1 - r - a)/(1 - r)]^2\cdot\{F_{\mathrm{eff}}/F_{\mathrm{ref}}\} \end{aligned}$$

Limits to Performance: XI

Optimizing Performance

In a background limited situation, one can characterize the optimal performance of an FPI by the product of the finesse times the transmission:

$$\mathrm{NEP} \propto T^{-1/2}\cdot R^{-1/2} \propto (F\cdot T)^{-1/2}$$

Therefore, we would wish to maximize the product:

$$\mathbf{Q \equiv F\cdot T}$$

Now: $T_{\max} \approx \{1 - (2aF)/(\pi\sqrt{r})\}$

$\Rightarrow Q \approx \{1 - (2aF)/(\pi\sqrt{r})\}\cdot F$

1

$\Rightarrow dQ/dF \approx 1 - (4aF)/(\pi\sqrt{r}) = 0$

For example: $F_{\mathrm{optimal}} \approx \pi/4a \approx 80$ for $a = 1\%$

$\Rightarrow r \approx 96\%$