Ideal Photon Detectors

Astronomy 525

Lecture 16

Outline

- Radiation Transport: Terminology
- Detectors Attributes and Performance Measures
- Electrical Bandwidth
- Ideal Photon Detector
  - Noise for a Photodetector
  - Corrections for photon occupancy

Radiation Transport: Terminology

- I_ν = specific intensity $\left[\frac{ergs}{\sec}\frac{1}{cm^2}\frac{1}{Hz}\frac{1}{sr}\right]$
  - energy from a given direction

$$I_\nu = h\nu c\frac{dn_\nu}{d\Omega}$$

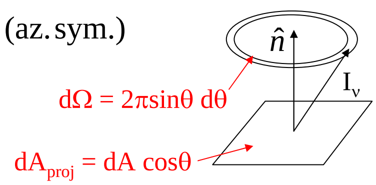
← number density of photons per unit frequency interval (#/cm³/Hz)

- F_ν = Flux density $\left[\frac{ergs}{cm^2 \sec Hz}\right]$ or $\left[\frac{ergs}{cm^2 \sec \mu m}\right]$

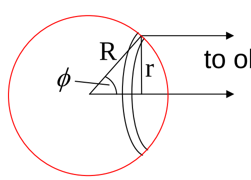
$$F_\nu = \int d\Omega\, I_\nu \cos\theta$$

$$\rightarrow \int d(\cos\theta)\, 2\pi\, I_\nu \cos\theta \quad \text{(az. sym.)}$$

$$\rightarrow \pi I_\nu \quad (I_\nu = \text{const.})$$



Flux from a star



$$f_\nu = \int d\Omega\, I_\nu \qquad (\cos\theta \approx 1)$$

$$d\Omega = \frac{dA_{proj}}{d^2} = \frac{2\pi\, r dr}{d^2} = -\frac{2\pi\, R^2 \cos\phi\ d(\cos\phi)}{d^2}$$

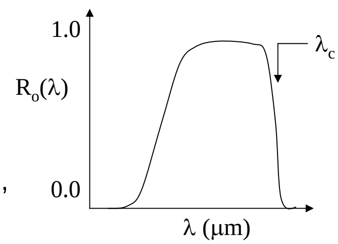
$$\Rightarrow f_\nu = 2\pi\frac{R^2}{d^2}\int_0^1 I_\nu(0,\mu)\mu d\mu \qquad (\mu = \cos\phi)$$

$$= \frac{R^2}{d^2}F_\nu \qquad \boxed{\Rightarrow f_\nu = \frac{R^2}{d^2}\pi B_\nu \quad (if\ \ I_\nu = B_\nu)}$$

- f_ν = monochromatic flux see by an observer
  - observed flux density
- f = flux seen by an observer

Attributes of Detectors

- Responsivity, R_o [amps/Watt]
  - ratio of output current to input power (due to photons)
  - gives no information about the noise properties of a detector, hence does not indicate sensitivity
  - measure at constant (DC) power
  - for one electron per photon: $R_o = e/h\nu = 0.81\ \lambda\ (\mu m)$
- Spectral Response, $R_o(\lambda)$
  - wavelength response of R_o



- Frequency Response, $R(\lambda,f)$
  - response to a modulated signal, e.g. chopped radiation

Attributes (cont'd)

- Quantum Efficiency, η (QE)
  - Average number of electrons generated per photon
  - Fraction of photons absorbed for bolometers
- Detective Quantum Efficiency, (DQE)
  - Square of output S/N relative to input S/N ratio
- Dark Current, i_d
  - Detector output with no photons falling on the device
- Read Noise, R_N (RN)
  - Detector noise w/ no input photons
  - Usually independent of integration time

Measures of Detector Performance

- NEP - Noise Equivalent Power ($W/Hz^{1/2}$)
  - The input power that produces an output signal-to-noise ratio of unity in a specified electrical bandwidth, Δf.
  - Often Δf = 1 Hz then the NEP is the minimum signal detectable in a ½ second measurement.
- NEF – Noise Equivalent Flux ($W/m^2/Hz^{1/2}$)
  - $NEF = NEP/(A_{tel}\eta_{atm}\eta_{tel})$
- D* - Specific Detectivity ($cm\text{-}Hz^{1/2}/W$) $D^* = \frac{\sqrt{A\Delta f}}{NEP}$
- NEFD - Noise Equivalent Flux Density
  - Input flux that produces a signal-to-noise ratio of unity in ½ second measurement
  - e.g. NEFD = 10 mJy (1 Jy = 10^{-26} W m^{-2} Hz^{-1})

Electrical Bandwidth

- We wish to characterize the electrical bandwidth of the system
- If a detection system responds uniformly to modulation frequency between f_1 and f_2 and no response outside, then the bandwidth is:

$$\Delta f = f_2 - f_1$$

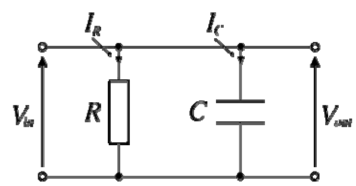
- Otherwise, if the response function, R(f) varies continuously then the equivalent bandwidth is:

$$\Delta f = \int_0^\infty \left| \frac{R(f)}{R_{\max}} \right|^2 df$$

where $R_{\max} = \max(R(f))$

Frequency Response

- One can characterize the response function of a detector by specifying the dependence of its output on the frequency of a sinusoidally varying input photon power.



- The frequency response can be limited by many factors in the detector system
  - However, most factors can be modeled as RC circuits where the R and C are in parallel
  - RC circuits have an exponential time rise response, so that charge deposited on the capacitor bleeds off through the resistor with an exponential time constant, τ_{RC} = RC

Frequency Response

- If we input a voltage pulse to the system: $v_{in}(t) = v_o\delta(t)$, then the output voltage, as observed with an oscilloscope will have the form:

$$v_{out}(t) = \begin{cases} 0, & t < 0 \\ \frac{v_0}{\tau_{RC}} e^{-t/\tau_{RC}}, & t \geq 0 \end{cases}$$

- The same event can be viewed in terms of the effect of the circuit on the input frequencies instead of in the time dependence of the voltage
- To do so, we take a Fourier transform of the input and output voltages to change to the frequency domain

$$V_{in}(f) \equiv FT\{v_{in}(t)\} = \mathrm{FT}\{v_0\delta(t)\}$$

Frequency Response

- The $\delta(t)$ function has contributions from all frequencies at equal strength, so that:

$$V_{in}(f) = v_0 \int_{-\infty}^{\infty} \delta(t) e^{-2\pi i f t} dt = v_0$$

- Since the input, $V_{in}(f)$ = constant, any deviations from a flat spectrum in the output must be due to the action of the circuit, i.e. the output spectrum yields the frequency response of the circuit directly:

$$V_{out}(f) = \int_{-\infty}^{\infty} v_{out}(t) e^{-2\pi i f t} dt = v_0 \left[\frac{1 - 2\pi i f \tau_{RC}}{1 + (2\pi f \tau_{RC})^2} \right]$$

Frequency Response

- The imaginary part is just a phase shift, which we ignore for now.
- We are therefore just concerned with the amplitude of the frequency response which is given by:

$$|V_{out}(f)| = \left(V_{out} V_{out}^*\right)^{1/2} = \frac{v_0}{\left(1 + (2\pi f \tau_{RC})^2\right)^{1/2}}$$

Frequency Response

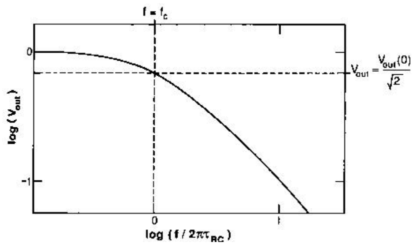


Figure 1.6. Frequency response of an *RC* circuit. The cutoff frequency is also indicated.

$$|V_{out}(f)| = \frac{V_0}{\left(1+(2\pi f\tau_{RC})^2\right)^{1/2}}$$

- The frequency response can be characterized by a cutoff frequency: $f_c = \dfrac{1}{2\pi\tau_{RC}}$
- At which the amplitude drops to 1/sqrt(2) of its value at f =0

$$|V_{out}(f_c)| = \frac{1}{\sqrt{2}}|V_{out}(0)|$$

EBW: Exponential Decay

- For such a system, with exponential decay time, τ, the response function, $R(f)$ is therefore:

$$R(f) = \frac{R_0}{1+2\pi i f\tau}$$

Or

$$\Delta f = \int_0^\infty \frac{df}{1+(2\pi f\tau)^2} = \frac{1}{4\tau}$$

Where the R_0 is the DC responsivity

E-BW: Integrator over time T

- For a system that integrates over a time, T, the response is:

$$v_{out}(t) = \begin{cases} 1 & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$

- The frequency response is:

$$R(f) = \int_{-T/2}^{T/2} \frac{R_0}{T} e^{-2\pi i f t}\, dt = R_0 \frac{\sin \pi f T}{\pi f T}$$

- The electrical bandwidth is then:

$$\Delta f = \int_0^{\infty} \left[\frac{\sin \pi f T}{\pi f T} \right]^2 df = \frac{1}{2T}$$

Ideal Photon Detector

- No output current in the absence of incident power
- No noise except that due to the randomness of emission times of photoelectrons
- Let P be the power falling onto the detector
  - with quantum efficiency, η, in a small bandwidth, $\Delta\nu$.
- Assume emission events are probabilistic in the sense that they are uncorrelated and occur at an average rate r, given by:

$$r = \frac{\eta P}{h\nu}$$

Ideal Photon Detector: Shot Noise

- The average number of photoevents occurring in a time T is: $\overline{N} = rT \qquad \left(\text{let } N_a \equiv \overline{N}\right)$
- The actual number of events will fluctuate around N_a for any one particular interval of length T.
- The probability, $P(N)$, that in any one such interval exactly N photoevents occur is given by the Poisson probability distribution:

$$P(N) = \frac{\overline{N}^N}{N!} e^{-N}$$

Applicability of Poisson Distribution

- Divide the time interval T into n segments. The average number of photons per segment is $\overline{N}/n$, where $\overline{N}$ is the average number of photon events for the time interval T.
- For sufficiently large n, $\overline{N}/n << 1$, so that $\overline{N}/n$ can be interpreted as the probability that one photoevent occurs in a given segment.
- The probability that exactly N events occur in the total interval T is given by the binomial distribution

$$P_n(N) = \frac{n!}{N!(n-N)!}\left[\frac{\overline{N}}{n}\right]^N \left[1 - \frac{\overline{N}}{n}\right]^{n-N}$$

Poisson Distribution

Combination of n things taken N at a time – the total number of ways that N indistinguishable events can occur in n segments

$$P_n(N) = \frac{n!}{N!(n-N)!}\left[\frac{\overline{N}}{n}\right]^N \left[1-\frac{\overline{N}}{n}\right]^{n-N}$$

Probability that photoevents do not occur in the remaining n-N segments

Probability that photoevents occur in N specific segments

- Rewriting gives

$$P_n(N) = \frac{1(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{N+1}{n})}{N!}\overline{N}^N\left[1-\frac{\overline{N}}{n}\right]^{n-N}$$

- Taking the limit as $n \to \infty$, we arrive at the Poison distribution

$$P(N) = \lim_{n\to\infty} P_n(N)$$

$$= \frac{\overline{N}^N}{N!}\lim_{n\to\infty}\left[1-\frac{\overline{N}}{n}\right]^{n-N} \quad \Rightarrow P(N) = \frac{\overline{N}^N}{N!}e^{-\overline{N}}$$

Poisson Probability Distribution

- Figure 8.4 in Boyd – the Poisson probability distribution
- Note how has the average number of photons gets large, the width of the distribution (noise) approaches root(N)

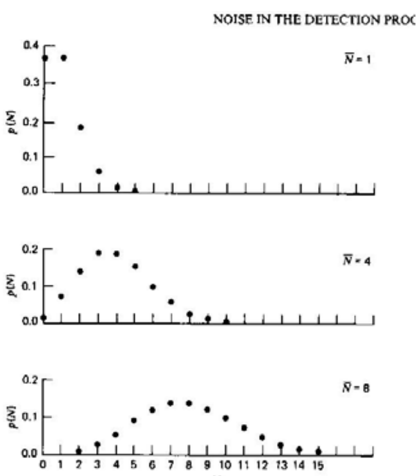


Figure 8.4. Poisson probability distribution.

Properties of Poisson Dist'n

- Normalization: $$\sum_{N=0}^{\infty} P(N) = e^{-\bar{N}} \sum_{N=0}^{\infty} \frac{\bar{N}^N}{N!} = e^{-\bar{N}} e^{\bar{N}} = 1$$
- One can show:
  - Expectation value: $$\langle N \rangle = \sum_{N=0}^{\infty} N\, P(N) = \bar{N}$$
  - Variance: $$\overline{(\Delta N)^2} \equiv \overline{(N-\bar{N})^2} = \bar{N}$$
  - RMS noise: $$\Delta N_{rms} \equiv \sqrt{\overline{(\Delta N)^2}} = \bar{N}^{1/2}$$
  - Signal to noise ratio: $$\frac{S}{N} \equiv \frac{\bar{N}}{\Delta N_{rms}} = \bar{N}^{1/2}$$ $$\propto T^{1/2}, \text{ since } \bar{N} = rT)$$

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Noise for a Photodetector

- The photoevents give rise to a photocurrent (and likewise noise) in a photon detector.
- Suppose a photon detector is characterized by an averaging time T, the average current is:

$$\bar{i} \equiv \frac{e\bar{N}}{T}$$

- The noise in the photocurrent is:

$$i_N^2 \equiv \overline{(\Delta i)^2} \equiv \overline{(i-\bar{i})^2} = \frac{e^2}{T^2}\overline{(N-\bar{N})^2} = \frac{e^2\bar{N}}{T^2}$$

so that

$$i_N^2 = \frac{e\bar{i}}{T} = 2e\bar{i}\Delta f \qquad (\text{using } \Delta f = 1/2T)$$

White noise - constant noise power per unit frequency interval.

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Signal-Limited Detection

- Suppose P_S is the signal power falling onto an ideal detector of quantum efficiency, η, the signal current is:

$$i_S = \frac{\eta e P_S}{h\nu} \qquad \text{w/ noise} \quad i_N = \sqrt{2 e i_S \Delta f} = \sqrt{\frac{2\eta e^2 P_S \Delta f}{h\nu}}$$

The signal-to-noise ratio is then

$$\frac{S}{N} = \frac{i_S}{i_N} = \sqrt{\frac{\eta P_S}{2h\nu\Delta f}} \qquad \Rightarrow NEP \equiv P_S\big|_{S/N=1} = \frac{2h\nu\Delta f}{\eta}$$

$$\Rightarrow NEP = \frac{h\nu}{\eta T}$$

Minimum detectable power (or NEP) will produce on average one photo-detection per measurement time T.

Background Limited Instrument Performance (BLIP)

- In addition to P_S there may be an unwanted background power, P_B. The signal and noise currents are now:

$$i_S = \frac{\eta e P_S}{h\nu} \qquad i_N = \sqrt{\frac{2\eta e^2 (P_S + P_B)\Delta f}{h\nu}}$$

The signal-to-noise ratio is then

$$\frac{S}{N} = \sqrt{\frac{\eta P_S^2}{2h\nu\Delta f (P_S + P_B)}}$$

BLIP occurs for $P_B >> P_S$, such as looking through the atmosphere in the infrared.

$$\Rightarrow NEP = \sqrt{\frac{2h\nu P_B \Delta f}{\eta}}$$

BLIP

Point Source Sensitivity

- Letting $t = 1/(2\Delta f)$ = integration time and τ_a, τ_i = atmospheric, instrument transmission. For BLIP

$$P_S = \frac{S}{N}\sqrt{\frac{h\nu P_B}{\eta t}}$$

Now for a point source

$$P_S = f_\lambda \Delta\lambda\, A_T \tau_a \tau_i$$

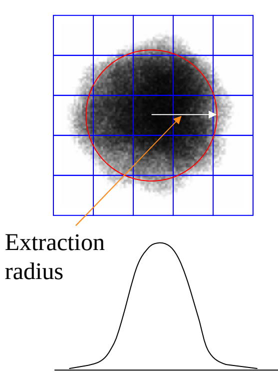
f_λ = ergs/cm²/s/μm
A_T = area of telescope.

$$\Rightarrow f_\lambda = \frac{S}{N}\frac{1}{\Delta\lambda\, A_T \tau_a \tau_i}\sqrt{\frac{h\nu P_B}{\eta t}}$$

Flux density to achieve a given S/N ratio in time t for BLIP.

Point Source Extraction

- How do we extract the photo-electrons for a point source?



Options

- Add-up signal in pixels that contain the source.
- Fit point spread function (PSF) to source (better)

- Optimizing extraction radius to get best S/N ratio (if not using PSF).
- Don't have to get "all" the flux - but must account for this in calibration and S/N calculations.

Point Source Sensitivity (cont'd)

- Assume background emission is extended and "thermal":

$$P_B = \varepsilon_\lambda B_\lambda(T)\,\Delta\lambda\, A_T \Omega \tau_i$$

ε = emissivity
A_T = telescope area
Ω = solid angle for extracting source

- If there are more sources of background, add them to get P_B.
- Putting the above in for P_B gives

$$f_\lambda = \frac{S}{N}\frac{1}{\tau_a}\sqrt{\frac{h\nu\,\varepsilon_\lambda B_\lambda}{\eta\,\tau_i\,t}\frac{\Omega}{\Delta\lambda\, A_T}}$$

The Complete Story: Photons are Bosons

- The probability distribution function for getting n photons per mode, that is, the probability that n photons are excited in a mode of angular frequency ω is given by:

$$p(n) = e^{-n\hbar\omega/kT} \Big/ \sum_{n=0}^{\infty} e^{-n\hbar\omega/kT}$$

- n is called the mode occupation number, and the energy of the state is:

$$E = n\hbar\varpi \qquad n = 0, 1, 2, \ldots$$

- Summing the series gives:

$$p(n) = \left(1 - e^{-\hbar\omega/kT}\right)e^{-n\hbar\omega/kT}$$

Bose-Einstein Probability Distribution

- The average occupation number is:

$$\bar{n} = \sum_{n=0}^{\infty} n p(n) \quad \Rightarrow \quad \bar{n} = \frac{1}{e^{h\nu/kT} - 1}$$

- We could then write:

$$p(n) = \frac{\bar{n}^n}{(1+\bar{n})^{1+n}}$$

Bose-Einstein probability distribution

- Now

$$\overline{(\Delta n)^2} = \overline{n^2} - \bar{n}^2 \qquad \text{where} \quad \overline{n^2} = \sum_{n=0}^{\infty} n^2 p(n)$$

Noise in Bose-Einstein Distribution

- So we have

$$\overline{n^2} = \sum_{n=0}^{\infty} n^2 p(n) = (1-e^{-x})\frac{d^2}{dx^2}\sum_{n=0}^{\infty} e^{-nx} \qquad \text{where} \quad x = \frac{h\nu}{kT}$$

$$= (1-e^{-x})\frac{d^2}{dx^2}\frac{1}{1-e^{-x}}$$

$$= \frac{e^x + 1}{(e^x - 1)^2}$$

$$\Rightarrow \quad \overline{n^2} = 2\bar{n}^2 + \bar{n}$$

- The rms dispersion becomes:

$$\Delta n_{rms} = \sqrt{\overline{(\Delta n)^2}} = \sqrt{\bar{n}(\bar{n}+1)}$$

for $h\nu >> kT$, i.e. $n << 1$

$$\Delta n_{rms} = \sqrt{\bar{n}}$$

for $h\nu << kT$, i.e. $n >> 1$

$$\Delta n_{rms} = \bar{n}$$

Photon "bunching" causes increased dispersion

Noise in Ideal Detector

- So we have $(\Delta n)^2_{rms} = \bar{n}(\bar{n}+1)$
- Now consider the power falling onto a detector

$$P_B = A\Omega\,\Delta\nu\,B_\nu(T) = A\Omega\,\Delta\nu\,\frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$
$$= h\nu\,\bar{n}\,r$$

where

$$r = 2\frac{A\Omega}{\lambda^2}\Delta\nu$$

r is the rate at which field modes intersect the detector

- The fluctuations in the photon number occur over the coherence time of the radiation field, $\tau_c \sim (\Delta\nu)^{-1}$. (see Boyd)

Transmission and QE effects

- However, the sampling time is much longer than the coherence time. If, t, is the integration time, the we sample a total of rt modes.
- If we add together rt modes the noise is then

$$(\Delta N)^2_{rms} = rt(\Delta n)^2_{rms} = rt\,\bar{n}(\bar{n}+1)$$
$$= \frac{rt\,P_B}{h\nu\,r}(1+\bar{n})$$

- If the detector has quantum efficiency, η, and optical transmission, τ, and the emissivity of the background is, ε, then the number of modes "reaching" the detector and producing photoelectrons is reduced by the product of the factors, i.e. $\bar{n} \rightarrow \eta\varepsilon\tau\bar{n}$

Signal-to-Noise Ratio

- We then have

$$(\Delta N)^2_{rms} = \frac{P_B\,\eta\varepsilon\tau\,t}{h\nu}\left(1+\eta\varepsilon\tau\,\bar{n}\right)$$

- The number of signal electrons generated will be

$$N_S = \frac{\eta\tau(1-\varepsilon)P_S}{h\nu}t$$

- The signal to noise is then given by

$$\frac{S}{N} \equiv \frac{N_S}{(\Delta N)_{rms}}$$

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Signal detection with B-E Statistics

- Putting it all together gives

$$NEP = \frac{1}{1-\varepsilon}\sqrt{\frac{\varepsilon\,P_B h\nu}{\eta\tau\,t}\left(1+\eta\varepsilon\tau\,\bar{n}\right)}$$

where

$$P_B = A\Omega\,\Delta\nu\,B_\nu(T) \qquad \& \qquad \bar{n} = \frac{1}{e^{h\nu/kT}-1}$$

- Note that for $h\nu >> kT$ the Bose-Einstein correction factor is unimportant. Let

$$x \equiv \frac{h\nu}{kT} = \frac{14388}{\lambda(\mu m)T}$$

for T = 290 K; $x = 49.6/\lambda(\mu\text{m})$

So the dividing line is ~ 50 µm for whether to worry about the extra factor (and depends upon η, ε, and τ.

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