Signal Detection

Astronomy 525

Lecture 17

Outline

- Detection Limits & Scaling Laws
  - Point and Extended Sources
  - Spectral Lines
- At the telescope
  - CCD Example
  - Choosing your extraction aperture
- Photons as bosons
  - Bose-Einstein statistics
  - Point source sensitivity
  - Additional complications
  - Putting it all together

Signal-to-Noise Ratio

- The signal-to-noise ratio is on a source with S collected photoelectrons is:

$$\frac{S}{N} = \frac{S}{\left[{N_S}^2 + {N_B}^2 + {N_d}^2 + R_N^2\right]^{1/2}}$$

- Since the (uncorrelated) noises add in quadrature
- Thus, solving for S and plugging in gives

$$\frac{\eta G \tau \tau_w P_S}{h\nu} t = \frac{S}{N}\left[\frac{\beta G^2 \eta \tau (\tau_w P_S + P_B)}{h\nu} t + \beta_d G_d \bar{i}_d t + R_N^2\right]^{1/2}$$

- If one of the noise terms dominates the others then the system is called: background, dark current or read noise limited depending on which dominates.
- The best condition is signal-noise-limited

τ_w = (warm) transmission
τ = instrument transmission
η = detector responsivity
G = Gain
β = Gain dispersion (excess noise)
R_N = read noise
N_x = Noise due to x

Point Source Sensitivity (cont'd)

- BLIP (Background Limited Infrared Performance) is defined as the background dominating all other noise sources.
- Assuming the background emission is extended and "thermal"

$$P_B = \varepsilon_\lambda B_\lambda(T)\,\Delta\lambda\, A_T \Omega \tau$$

ε = emissivity
A_T = telescope area
Ω = solid angle for extracting source

- If there are more sources of background, add them to get P_B. Putting the above in for P_B gives

$$f_\lambda = \frac{S}{N}\frac{1}{\tau_w}\sqrt{\frac{h\nu\,\varepsilon_\lambda B_\lambda}{(\eta/\beta)\tau t}\frac{\Omega}{\Delta\lambda\, A}}$$

- BLIP is the best you can do (unless your source is bright enough to reach the signal-noise-limited regime)

Scaling Laws for Point Sources

- For diffraction limited performance:

$$\left(1.22\frac{\pi}{4}\right)^2 = 0.92$$

$$A_T\Omega = 0.92\lambda^2 \quad \Rightarrow \quad f_\lambda \propto \frac{1}{D^2\sqrt{t}} \qquad t \propto \left[\frac{S}{N}\right]^2 \frac{1}{f_\lambda^2 D^4}$$

Diffraction limited beam size

- For a fixed beam size (e.g. seeing limited):

$$\Omega = \frac{\pi}{4}\theta_B^2 \quad \Rightarrow \quad f_\lambda \propto \frac{\theta_B}{D\sqrt{t}} \qquad t \propto \left[\frac{S}{N}\right]^2 \frac{\theta_B^2}{f_\lambda^2 D^2}$$

Fixed beam size

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Extended Source Sensitivity

- For an extended source on the sky. Let

$$f_\lambda = I_\lambda \Omega$$

I_λ = specific intensity (ergs/cm^2/sec/sr/μm)

- Then

$$I_\lambda = \frac{S}{N}\frac{1}{\tau_a}\sqrt{\frac{h\nu\,\varepsilon_\lambda B_\lambda}{\eta\,\tau_i\,t}\frac{1}{\Delta\lambda\, A_T\Omega}}$$

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Scaling Laws for Extended Sources

- For diffraction limited performance:

$$A_T\Omega = 0.92\lambda^2 \quad \Rightarrow \quad I_\lambda \propto \frac{1}{\sqrt{t}} \qquad t \propto \left[\frac{S}{N}\right]^2 \frac{1}{I_\lambda^2}$$

Diffraction limited beam size

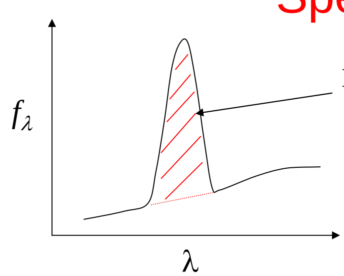
Independent of D!

- For a fixed beam size:

$$\Omega = \frac{\pi}{4}\theta_B^2 \quad \Rightarrow \quad I_\lambda \propto \frac{1}{\theta_B D\sqrt{t}} \qquad t \propto \left[\frac{S}{N}\right]^2 \frac{1}{\theta_B^2 I_\lambda^2 D^2}$$

Fixed beam size

Spectral Lines



Line flux = area (f)
= (ergs/cm^2/sec)

- For best sensitivity
  - Narrow spectral bandpass to remove extraneous flux without reducing line flux
- The integrated flux will be roughly: $f = f_\lambda \Delta\lambda$ so that the sensitivity is now:

$$f = \frac{S}{N}\frac{1}{\tau_a}\sqrt{\frac{h\nu\,\varepsilon_\lambda B_\lambda}{\eta\,\tau_i\,t}\frac{\Delta\lambda\,\Omega}{A_T}}$$

narrower spectral BW
$\Rightarrow$ better sensitivity,
unless line is resolved

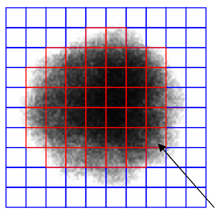
Point Sources: At the telescope

- Consider a CCD operating at a telescope.
- We'll use the measured parameters to estimate sensitivity (or integration time). We have:

$$Signal = St$$

$$Noise = \left(St + pS_B t + pR_N^2\right)^{1/2}$$

S = e-/sec from source
S_B = e-/sec/pixel from sky
p = number of pixels
R_N = read noise (e-)



sum over p pixels

$$\frac{S}{N} = \frac{St}{\left(St + pS_B t + pR_N^2\right)^{1/2}}$$

Optical Imaging Palomar

- Suppose we have a imaging CCD at Palomar with the following characteristics (the "old" COSMIC instrument).

CCD: 2048x2048 R_N: 11 e-
Scale: 0.2856 arcsec/pixel
20th mag. star: 637 e-/sec at R-band (S_{R20})
sky: 18 e-/sec/pix at R-band (B_R)

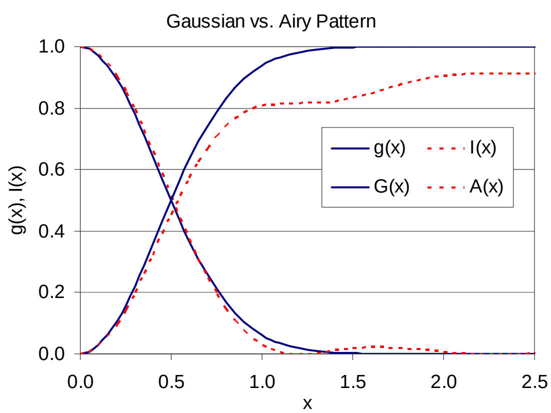
- Ignoring the read noise (okay for $t > 25$ sec), the time to obtain a give S/N at R-band on a source of magnitude m is

$$t_R = \left[\frac{S}{N}\right]^2 \frac{S_{R20}10^{-(m-20)/2.5} + pB_R}{S_{R20}^2 10^{-(m-20)/1.25}}$$

e.g. t_R = 430 seconds to get S/N = 5 for m = 25 with an extraction aperture of 10 pixels containing 50% of the flux.

What is Ω, the extraction beam?

- We are left with choosing the extraction beam (number of pixels used to extract the source flux). This is science choice dependent! For point sources, in the seeing limited case we might expect a Gaussian PSF while for diffraction limited observation the Airy diffraction pattern will hold.



Plot of Gaussian and Airy diffraction pattern, and the encircled energy (weighted area integral) vs. *x*, where *x* is in units of FWHM for the Gaussian and λ/D for the Airy pattern.

The HWHM of the Airy pattern is 0.51λ/D (and of course 0.5 for the Gaussian).

The Airy pattern is for D_{obscur}/D_{tel} = 0.12 or 1.44% obscured area.

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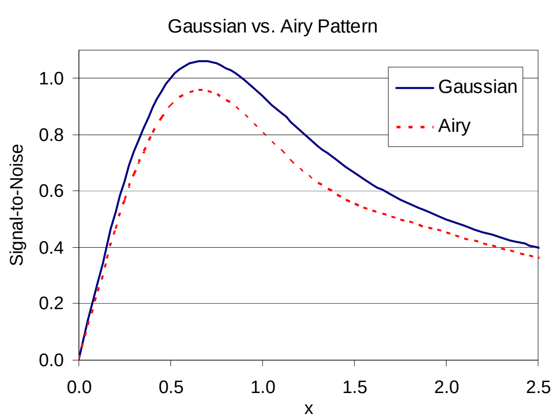
Optimum Signal-to-Noise Ratio

- For BLIP observations the signal-to-noise ratio will vary as the encircled energy divided by *x* (since the background power will scale as x^2 and the noise scales as the square root of the power - However at longer wavelengths this is no longer true because Bose-Einstein statistics become important).



Plot of relative signal-to-noise ratio for a Gaussian and Airy diffraction pattern vs. *x*, where *x* is in units of FWHM for the Gaussian and λ/D for the Airy pattern.

The maximum S/N occurs at *x* = 0.673 and 0.660 for the Gaussian and Airy pattern respectively. Choosing *x* = 0.5 for the extraction radius causes a 6% decrease in S/N.

The Airy pattern is for D_{obscur}/D_{tel} = 0.12 or 1.44% obscured area.

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Choosing Ω

- Our first guess for an extraction diameter for a diffraction limited beam might have been 1.22λ/D which would give

$$A_T\Omega = \frac{\pi}{4}D_T^2\frac{\pi}{4}\left[1.22\frac{\lambda}{D_T}\right]^2 = 0.92\lambda^2$$

- From the results in optimizing S/N a better choice of the extraction diameter is 1.294×FWHM = 1.32 λ/D_T. Now we have

$$A_T\Omega = \frac{\pi}{4}D_T^2\frac{\pi}{4}\left[1.32\frac{\lambda}{D_T}\right]^2 = 1.08\lambda^2$$

- The table below shows how the fractional flux and relative S/N change with extraction aperture diameter for the Airy diffraction pattern

θ (λ/D)	Flux Frac.	S/N
1.00	0.45	0.90
1.25	0.59	0.92
1.345	0.64	0.96

θ (λ/D)	Flux Frac.	S/N
1.50	0.71	0.94
1.75	0.78	0.89
2.00	0.81	0.81

Choosing Ω

- The table below summarizes the results for Gaussian and diffraction limited PSFs with the addition of a final column for the fraction of the flux in the extracted beam.
- Nicely, the ratio of the optimum extraction radius to HWHM is about the same for both so adopting an average value will result in only a few percent error.

PSF	HWHM	Opt. S/N radius	Opt S/N over HWHM	Fraction of Flux in Extraction
Gaussian	0.500	0.673	1.346	0.715
Airy	0.510	0.660	1.294	0.632

The units for the HWHM and optimum extraction radius are FWHM for the Gaussian and λ/D for the Air diffraction pattern.

- A combined expression for the extraction diameter (when diffraction and Gaussian terms are both present) might look like

$$d_{ext} = \sqrt{(1.294FWHM_A)^2 + (1.346FWHM_G)^2}$$
$$= 1.32\sqrt{(\lambda/D)^2 + (1.02FWHM_G)^2}$$

Boson Effects: Signal-to-Noise Ratio

- Including Bose-Einstein statistics

$$\sigma_B^2 \equiv N_B^2 = \frac{P_B \eta \varepsilon \tau t}{h\nu}\left(1+\eta\varepsilon\tau\bar{n}\right)$$

correction term for the fact that photons are Bosons

- The number of signal electrons generated will be

$$S = \frac{\eta \tau \tau_w P_S}{h\nu} t \qquad \tau_w = 1 - \varepsilon$$

- The signal to noise is then given by

$$\frac{S}{N} \equiv \frac{S}{N_B}$$

ε = emissivity of warm background (atmosphere and telescope combined)

Signal detection with B-E Stats

- Thus we have

$$P_S = \frac{S}{N}\frac{1}{1-\varepsilon}\sqrt{\frac{\varepsilon P_B h\nu}{\eta\tau t}\left(1+\eta\varepsilon\tau\bar{n}\right)}$$

where

$$P_B = A_T \Omega \Delta\nu B_\nu(T) = 2\frac{A_T\Omega}{\lambda^2}\Delta\nu h\nu\bar{n} \qquad \& \qquad \bar{n} = \frac{1}{e^{h\nu/kT}-1}$$

- Thus we can write:

$$P_S = \frac{S}{N}\frac{1}{1-\varepsilon}\frac{h\nu}{\eta\tau}\sqrt{\bar{n}_r\left(1+\bar{n}_r\right)}\sqrt{\frac{\Delta\nu}{t}2\frac{A_T\Omega}{\lambda^2}} \qquad \bar{n}_r = \eta\varepsilon\tau\bar{n}$$

The factor of two enters since we have both polarizations. Calling N_p the number of polarization measured (1 or 2), we have for point sources

$$P_S = A_T f_\nu \Delta\nu \frac{N_p}{2}$$

We reference to the unpolarized flux.

Sensitivity with B-E Stats

- Putting it all together gives

$$f_\nu = \frac{S}{N}\frac{1}{1-\varepsilon}\frac{h\nu}{\eta\tau}\frac{2}{A_T}\sqrt{\frac{\bar{n}_r(1+\bar{n}_r)}{N_p\Delta\nu}}\sqrt{\frac{A_T\Omega}{\lambda^2}}\frac{1}{\sqrt{t}}$$

Where for diffraction limited performance

$$A_T\Omega \propto \lambda^2$$

- Note that for $h\nu >> kT$ the Bose-Einstein correction factor is unimportant. Let

$$x \equiv \frac{h\nu}{kT} = \frac{14388}{\lambda(\mu m)T}$$

for T = 290 K; $x = 49.6/\lambda(\mu\text{m})$

So the dividing line is ~ 50μm for whether to worry about the extra factor (and depends upon η, ε, and τ.

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Additional Complications

- Multiple background contributions
  - The derivation on the previous page assumed a single background source with emissivity ε [and transmission (1-ε)].
  - Emission is also usually contributed by the telescope (and other sources) which may or may not be at the same temperature as the atmosphere.
- Lost light
  - In the radio and submm parts of the spectrum, typically the surface roughness of the "dish" is large enough to scatter light from the source out of the beam (but the background will be unchanged because light from the background is also scattered into the beam)
- Point source extraction
  - The optimal extraction of a point source will not include the entire PSF – so we must account for this.
  - Pixilation intrinsically widens the source PSF. To first order this can be modeled by increasing the effective extraction size (by adding the pixel angular size in quadrature with the nominal extraction beam size).

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Additional Complications (continued)

- Detector Noise
  - Detectors may have read noise, generation-recombination noise, or other sources of noise but we will ignore these for now.
- Chopping / Nodding
  - In the infrared/submm a source is move rapidly (chopped) between two sky positions. The difference is taken to remove atmospheric variation, telescope offsets, etc.
  - This differencing adds the noises, thus decreasing the sensitivity by sqrt(2).
  - Additionally if the source is moved off the detector (array) for the second sky position, a factor of two in time is also lost resulting in another factor of sqrt(2) change in sensitivity.
  - These factors are not included in what follows.

Sensitivity with B-E Stats

- The light loss is characterized by the Ruze factor, g_R, which is given by

$$g_R = e^{-(4\pi s_{rms}/\lambda)^2} \qquad s_{rms} = \text{rms surface roughness}$$

- Let the telescope have emissivity, ε_T and temperature T_T, we then have

$$\bar{n}_r = \eta\tau\left[\frac{\varepsilon_A}{e^{h\nu/kT_A}-1}+\frac{\varepsilon_T}{e^{h\nu/kT_T}-1}\right]$$

where the atmospheric component is now explicitly labeled and attenuation by the telescope is neglected (normally a 2nd order effect here).

- Other backgrounds (emissivity sources) can be added in a similar way.
- Finally letting g_f be the fraction of the source flux in the extraction beam (typically ~ 70%) and τ_T the telescope transmission, we have

$$f_\nu = \frac{(S/N)}{(1-\varepsilon_A)\tau_T}\frac{1}{g_f g_R}\frac{h\nu}{\eta\tau}\frac{2}{A_T}\sqrt{\frac{\bar{n}_r(1+\bar{n}_r)}{N_p\Delta\nu}}\sqrt{\frac{A_T\Omega}{\lambda^2}}\frac{1}{\sqrt{t}}$$

Point Source Sensitivity

- In summary, the sensitivity for instrument on a telescope including the Bose-Einstein contribution is:

$$f_\nu = \frac{(S/N)}{\tau_A \tau_T} \frac{1}{g_f g_R} \frac{h\nu}{\eta\tau} \frac{2}{A_T} \sqrt{\frac{\overline{n}_r(1+\overline{n}_r)}{N_p \Delta\nu}} \sqrt{\frac{A_T \Omega}{\lambda^2}} \frac{1}{\sqrt{t}}$$

$$g_R = e^{-(4\pi s_{rms}/\lambda)^2} \qquad \overline{n}_r = \eta\tau\left[\frac{\varepsilon_A}{e^{h\nu/kT_A}-1} + \frac{\varepsilon_T}{e^{h\nu/kT_T}-1}\right]$$

S/N, *t*	= signal-to-ratio and integration time
ε_T, ε_A	= telescope and atmospheric emissivity
τ_T, τ_A	= telescope and atmospheric transmission (τ_A = 1 - ε_A)
T_T, T_A	= telescope and atmospheric temperature
τ, η, N_p	= instrument transmission, detector QE, number of polarizations (1 or 2)
A_T, g_f	= Telescope area and fraction of flux in extraction (~ 0.7 typically)
s_{rms}	= rms surface roughness

Chopping and Pixilation

- We can add a chopping degradation factor fairly easily since this directly affects the noise though a difference (sqrt(2)) or integration time on source (for a given wall clock time).
- As a first order assumption for "pixilation" noise assume that there are *p* pixels across the diffraction limit. Then

$$\Omega_{PB} = \Omega\left[1 + \frac{1}{p^2}\right]$$

  - Where Ω is the non-pixilated beam area. *p* = 2.0, 2.5, and 3.0 yields factors of 1.12, 1.08, and 1.05 change in sensitivity respectively.
  - Note that this formulation assumes that a set of randomly "dithered" images are combined so the effective beam is widened which is done to eliminate systematic.
  - This effect is negated with "perfect" pointing so that this is no "smearing" of the beam, however, quantization effects (finite size pixels) will still enter.

Point Source Sensitivity, again

- Including pixilation and chopping loses the point source sensitivity is

$$f_\nu = \frac{(S/N)}{\sqrt{t}} \frac{C_L}{g_f g_R} \frac{h\nu}{\tau_A \tau_T \eta \tau} \frac{2}{A_T} \sqrt{\frac{\bar{n}_r (1+\bar{n}_r)}{N_p \Delta\nu}} \sqrt{\frac{A_T \Omega}{\lambda^2}} \sqrt{1+\frac{1}{p^2}}$$

$$g_R = e^{-(4\pi s_{rms}/\lambda)^2} \qquad \bar{n}_r = \eta\tau \left[\frac{\varepsilon_A}{e^{h\nu/kT_A}-1} + \frac{\varepsilon_T}{e^{h\nu/kT_T}-1} \right]$$

S/N, t	= signal-to-ratio and integration time
ε_T , ε_A	= telescope and atmospheric emissivity
τ_T , τ_A	= telescope and atmospheric transmission (τ_A = 1 - ε_A)
T_T, T_A	= telescope and atmospheric temperature
τ , η, N_p	= instrument transmission, detector QE, number of polarizations (1 or 2)
A_T , g_f	= Telescope area and fraction of flux in extraction (~ 0.7 typically)
s_{rms} , C_L	= rms surface roughness and chopping loss (1, 1.414, or 2 typically)
p	= pixels across beam (= 2 for Nyquist sampling)