Noise

Astronomy 525

Lecture 21

Outline

- Spectral density
- Correlation function
- Weiner-Khintchine Theorem
- Campbell’s Theorem
- Carson’s Theorem
  - Spectral Density of Photon Noise
  - Systems with time constants
- Non-ideal Photon Detection
  - Johnson Noise

Noise: What is it?

- The rms deviation from the time averaged signal is a measure of the “noise” in the detection system.

$$\Delta v_{\rm rms} = \sqrt{\overline{\left(v(t) - \overline{v}\right)^2}}$$

time-averaged signal

signal

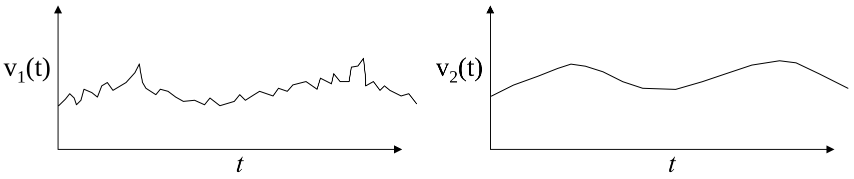
where

$$\overline{(\Delta v)^2} = (\Delta v_{\rm rms})^2 = \text{variance}$$

- This is not the whole story. It only gives us limited information on the signal properties.

Discussion follows Boyd

Noise: What is it?



- Consider two signals that have the same average and variance, as shown in the plots above. We have

$$\overline{v}_1 = \overline{v}_2 \qquad \& \qquad (\Delta v_1)_{\rm rms} = (\Delta v_2)_{\rm rms}$$

- But, the character (time scales) of the fluctuations are different because of the different frequency components in the noise.
- How do we quantitatively make a distinction?

Fourier Transform

- We can look at the harmonic content of the signal via the Fourier Transform:

$$\mathrm{V}(f) = \int_{-\infty}^{\infty} \mathrm{v}(\mathrm{t}) e^{-i2\pi f t} dt$$

The inverse transform gives back the signal

$$\mathrm{v}(\mathrm{t}) = \int_{-\infty}^{\infty} \mathrm{V}(f) e^{i2\pi f t} dt$$

- However, if v(t) is a stationary, stochastic function of time for $-\infty < t < \infty$, then v(t) is not square-integrable and V(f) does not exit.

stochastic: non-deterministic behavior (current state does not fully determine the next)
stationary: probability distribution is the same for all times (mean, variance, etc. are constant)

Spectral Density

- We can use a truncated signal $\mathrm{v_T(t)}$

$$\mathrm{v_T}(\mathrm{t}) = \begin{cases} \mathrm{v}(\mathrm{t}) & \mathrm{-T/2 < t < T/2} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathrm{V_T}(f) = \int_{-T/2}^{T/2} \mathrm{v}(\mathrm{t}) e^{-i2\pi f t} dt$$

- The spectral density, $S(f)$, is defined by (here < > indicates an ensemble average)

$$S(f) = \lim_{\mathrm{T}\to\infty} \left\langle \frac{1}{\mathrm{T}} \left| \mathrm{V_T}(f) \right|^2 \right\rangle$$

Spectral Density

- For most signals, we avoid performing the ensemble average and calculate $S(f)$ over some (perhaps very small) frequency interval

$$S(f) = \lim_{\mathrm{T}\to\infty} \frac{1}{\mathrm{T}} \left| \mathrm{V}_\mathrm{T}(f) \right|^2$$

This is the definition Boyd adopts for the spectral density (ergoticity)

- $S(f)$ = power per unit frequency interval that a voltage signal $V(t)$ could deliver to a 1 Ω resistor.

ergodic hypothesis: assume that the average of a process parameter over time and the average over the statistical ensemble are the same.

Variance from the Spectral Density

- Now

$$\overline{\mathrm{v}^2} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{v}^2(\mathrm{t})\,dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \mathrm{v}_\mathrm{T}^2(\mathrm{t})\,dt$$

- By Parseval’s theorem for the transform pair, x and X

$$\int_{-\infty}^{\infty} \left| x(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| X(f) \right|^2 df$$

Then using the fact that since $\mathrm{v}_\mathrm{T}(\mathrm{t})$ is real, the average power can be expressed as

$$\overline{\mathrm{v}^2} = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \left| \mathrm{V}_\mathrm{T}(f) \right|^2 df$$

$$= \int_{-\infty}^{\infty} S(f)\,df$$

The Correlation Function

- The limiting process required to compute $S(f)$

$$S(f)=\lim_{\mathrm{T}\to\infty}\frac{1}{\mathrm{T}}\left|\mathrm{V}_{\mathrm{T}}(f)\right|^{2}$$

is awkward to use in practice. As it turns out the correlation function can be related to the spectral density

- In general the correlation function looks like

$$C(t_1,t_2)=\langle \mathrm{v}(t_1)\mathrm{v}(t_2)\rangle$$

- Where the brackets indicate an ensemble average. We usually deal with signals that are ergodic (random signals with time average = an ensemble average)

Stationary Signals

- For stationary signals $C(t_1,t_2)$ must only depend on the difference $\tau = t_1 - t_2$. Thus

$$C(\tau)=\lim_{\mathrm{T}\to\infty} C_{\mathrm{T}}(\tau)$$

where

$$C_T(\tau)=\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{v}(t)\mathrm{v}(t+\tau)dt$$

- The correlation function is usually represented as

$$C(\tau)=\overline{\mathrm{v}(t)\mathrm{v}(t+\tau)} \qquad \longleftarrow \text{time average}$$

At $\tau = 0$,

$$C(0)=\overline{\mathrm{v}^2}$$

Properties of Correlation Function

- The correlation function is an even function of τ

$$C(-\tau)=\overline{\mathrm{v}(t)\mathrm{v}(t-\tau)}=\overline{\mathrm{v}(t+\tau)\mathrm{v}(t)}=C(\tau)$$

We also have

$$[\mathrm{v}(t)\pm \mathrm{v}(t+\tau)]^2 \geq 0$$

$$\Rightarrow \quad \mathrm{v}^2(t)+\mathrm{v}^2(t+\tau)\geq \pm 2\,\mathrm{v}(t)\mathrm{v}(t+\tau)$$

- and taking the time average of both sides give

$$\overline{\mathrm{v}^2} = \pm\overline{\mathrm{v}(t)\mathrm{v}(t+\tau)}$$

$$\Rightarrow \quad C(0)\geq |C(\tau)|$$

Perfect correlation with zero delay – can't beat that.

Wiener-Khintchine Theorem

- The W-K theorem states that $S(f)$ and $C(\tau)$ are a Fourier Transform pair. Consider

$$\int_{-\infty}^{\infty} C(\tau)e^{-i2\pi f\tau}d\tau = \tfrac{1}{T}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} dt\, \mathrm{v}_T(t)\mathrm{v}_T(t+\tau)e^{-i2\pi f\tau}$$

$$= \tfrac{1}{T}\int_{-\infty}^{\infty} dt\, \mathrm{v}_T(t)e^{i2\pi ft}\int_{-\infty}^{\infty} d\tau\, \mathrm{v}_T(t+\tau)e^{-i2\pi f(t+\tau)}$$

$$= \tfrac{1}{T}\mathrm{V}_T(-f)\mathrm{V}_T(f)$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} C(\tau)e^{-i2\pi f\tau}d\tau = \tfrac{1}{T}|\mathrm{V}_T(f)|^2$$

- This is a special case of the convolution theorem, which states that the Fourier transform of the convolution of two functions equals the product of the Fourier transforms.

Wiener-Khintchine Relations

- Letting T → ∞ $\Rightarrow \; S(f) = \int_{-\infty}^{\infty} C(\tau) e^{-i2\pi f\tau} d\tau$
- Taking the inverse transform gives

$$C(\tau) = \int_{-\infty}^{\infty} S(f) e^{i2\pi f\tau} df$$

- This and the previous equation are the Weiner-Khintchine relations. If $C(\tau)$ is an even function of τ then

$$S(f) = 2\int_0^{\infty} C(\tau) \cos 2\pi f\tau \, d\tau$$

- Which means $S(f)$ is an even function of f, hence

$$C(\tau) = 2\int_0^{\infty} S(f) \cos 2\pi f\tau \, df$$

- These two equations are another form of the W-K relations

Wiener-Khintchine Relations

- In practice we have $0 < f < \infty$, rather than $-\infty < f < \infty$. The spectral density of v(t) is then often expressed as

$$\overline{\mathrm{v}^2}(f) = 2S(f)$$

- We then have for the W-K relations

$$\overline{\mathrm{v}^2}(f) = 4\int_0^{\infty} C(\tau) \cos 2\pi f\tau \, d\tau$$

and

$$C(\tau) = \int_0^{\infty} \overline{\mathrm{v}^2}(f) \cos 2\pi f\tau \, df$$

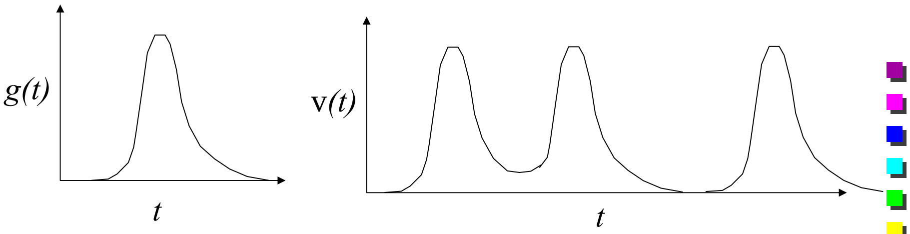
- And the mean square of v becomes

$$\overline{\mathrm{v}^2} = \int_{-\infty}^{\infty} S(f) df = 2\int_0^{\infty} S(f) df = \int_0^{\infty} \overline{\mathrm{v}^2}(f) df$$

Example: Identical Events

- Consider identical events occurring at random times (see Boyd, pg. 125-127)

$$\mathrm{v}_T = \begin{cases} \sum_{i=1}^{N} g(t-t_i) & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$



Average Signal and Event Rate

- The time average of v(t) is then

$$\overline{\mathrm{v}} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \sum_{i=1}^{N} g(t-t_i)\,dt$$

Which can be expressed as

$$\overline{\mathrm{v}} = r \int_{-\infty}^{\infty} g(t)\,dt \quad \text{where} \quad r = \left\langle \frac{N}{T} \right\rangle = \text{average rate of events}$$

$$\text{or} \qquad r = \lim_{T\to\infty} \frac{N}{T} \quad \text{(ergodic)}$$

- The average signal is just the event rate times the integrated amount in one event

Statistical Properties

- To look at the statistical properties of these events we look at the correlation function

$$C_T(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{v}(t)\mathrm{v}(t+\tau)dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\sum_{i=1}^{N} g(t-t_i)\sum_{i=1}^{N} g(t+\tau-t_i)\,dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} dt\left\{\sum_{i=1}^{N} g(t-t_i)g(t+\tau-t_i) + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} g(t-t_i)\,g(t+\tau-t_j)\right\}$$

- The first term in {} is just the correlation function of $g(t)$ with itself. Since no correlation exists between individual pulses we have

$$C_{\mathrm{v}}(\tau) = r\int_{-\infty}^{\infty} g(t)g(t+\tau)dt + \bar{\mathrm{v}}^2$$

Campbell's Theorem

- Using the variance in the signal as a measure of the noise

$$\overline{(\Delta\mathrm{v})^2} = \overline{[\mathrm{v}(t) - \bar{\mathrm{v}}]^2} = \overline{\mathrm{v}^2(t)} - \bar{\mathrm{v}}^2$$

Since $\overline{\mathrm{v}^2(t)} = C_{\mathrm{v}}(0)$ $\quad \overline{(\Delta\mathrm{v})^2} = r\int_{-\infty}^{\infty} g^2(t)dt$

- This expression and the earlier one for the average signal at known as Campbell's theorem, that is,

$$\boxed{\bar{\mathrm{v}} = r\int_{-\infty}^{\infty} g(t)dt} \quad \& \quad \boxed{\overline{(\Delta\mathrm{v})^2} = r\int_{-\infty}^{\infty} g^2(t)dt}$$

Carson's Theorem

- The spectral density can be computed from the correlation function $S(f)=r\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(t)g(t+\tau)dt+\bar{\mathrm{v}}^2\right)e^{-i2\pi f\tau}d\tau$

Proceeding as we did with the W-K theorem and using the fact that

$$\delta(f)=\int_{-\infty}^{\infty} e^{-i2\pi f\tau}d\tau \quad \text{where} \quad \int_{-\infty}^{\infty}\delta(f)e^{i2\pi f\tau}df=1$$

gives

$$\boxed{S(f)=r\left|G(f)\right|^2+\bar{\mathrm{v}}^2\delta(f)} \quad \text{where} \quad \boxed{G(f)=\int_{-\infty}^{\infty} g(t)e^{-i2\pi ft}dt}$$

- This is called Carson's theorem.
- The $\delta(f)$ term is the average power in the dc component of $G(t)$, i.e. $\delta(f)\neq 0$ only for $f=0$.
- The $G(f)$ term represents the noise in v(t) which is related to the spectral content of each random event.

Spectral Density of Photon Noise

- Recalling earlier we wrote $\overline{\mathrm{v}^2}(f)=2S(f)$
- Or we could write for the noise current due to photons

$$\overline{\mathrm{i}_{\mathrm{N}}^2}(f)=2S(f)$$

- An using our derived expression for the spectral density for random uncorrelated events we have

$$\overline{\mathrm{i}_{\mathrm{N}}^2}(f)=2r\left|G(f)\right|^2$$

- Where the dc term [$\delta(f)$] has been dropped since this is just the average current.

Spectral Density of Photon Noise

- If the pulses are unresolved

$$g(t) = e\delta(t) \quad \Rightarrow \quad G(f) = \int_{-\infty}^{\infty} g(t)e^{-i2\pi ft}\,dt = e$$

- And the current noise spectral density is

$$\overline{\mathrm{i}_{\mathrm{N}}^{2}}(f) = 2re^{2} = 2e\bar{i}$$

- The total noise power is obtained by multiplying by the electrical bandwidth, Δf

$$\overline{(\Delta i)^{2}} = \overline{\mathrm{i}_{\mathrm{N}}^{2}}(f)\Delta f = 2e\bar{i}\Delta f$$

- Note that this is “white noise”

Real System: Time constant

- In a real system we might expect a time constant, e.g. an exponential decay when a low pass filter is used

$$g(t) = \begin{cases} 0 & t < 0 \\ \dfrac{e}{\tau}e^{-t/\tau} & \mathrm{t} \geq 0 \end{cases} \quad \Rightarrow \quad \mathrm{G}(f) = \frac{e}{1 + i2\pi f\tau}$$

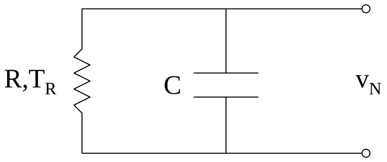
- The spectral density of the current noise is then

$$\overline{\mathrm{i}_{\mathrm{N}}^{2}}(f) = \frac{2e\bar{i}}{1 + (2\pi f\tau)^{2}}$$

- For $f \ll (2\pi\tau)^{-1}$ this approximates white noise again

Johnson Noise

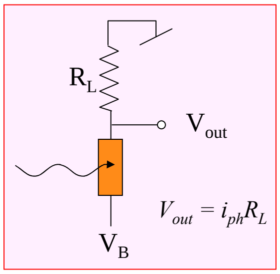
- Thermal agitation of electrons in a resistor cause noise



stored energy = $\frac{1}{2} C\overline{\mathrm{v}_{\mathrm{N}}^2}$

- Equipartition ⇒ average energy stored in our one dimensional classical system is equal to $kT_R/2$

$$\Rightarrow \qquad \overline{\mathrm{v}_{\mathrm{N}}^2} = \frac{kT_R}{C}$$



Correlation Function/Spectral Density

- The correlation function of the voltage fluctuations gives the transient behavior of the circuit

$$C(\tau) = \overline{\mathrm{v}(t)\mathrm{v}(t+\tau)} = \overline{\mathrm{v}(t)\mathrm{v}(t)}e^{-|\tau|/RC} = \overline{\mathrm{v}_N^2}e^{-|\tau|/RC}$$

- The spectral density of the voltage noise is then (slide 14)

$$\overline{\mathrm{v}_{\mathrm{N}}^2}(f) = 4\int_0^\infty C(\tau)\cos 2\pi f\tau\, d\tau$$

- So that

$$\overline{\mathrm{v}_{\mathrm{N}}^2}(f) = \frac{4\overline{\mathrm{v}_{\mathrm{N}}^2}RC}{1+(2\pi fRC)^2}$$

$$= 4\overline{\mathrm{v}_{\mathrm{N}}^2}RC \qquad \text{For } f << 1/RC$$

Johnson Noise: Result

- From the equipartition result

$$\boxed{\overline{\mathrm{v}_{\mathrm{N}}^2}(f) = 4kT_R R}$$

Johnson Noise

- Note: the is no C in the final result. Thus we can take C – 0, i.e. valid for all frequencies. This is another example of white noise. (This is only true for *hf* << *kT*)
- For Johnson noise limited detection

$$NEP = \frac{h\nu}{\eta e}\sqrt{\frac{4kT_R}{R}} \quad \Leftarrow$$

$$\mathrm{v}_{\mathrm{signal}} = i_{ph}R = \frac{\eta e P_S}{h\nu}R$$

$$\frac{S}{N} = \frac{\mathrm{v}_{\mathrm{signal}}}{\left(\overline{\mathrm{v}_{\mathrm{N}}^2}\right)^{1/2}} = \frac{\eta e P_S}{h\nu}R\frac{1}{\sqrt{4kT_R R}}$$

$NEP = P_s$ for $S/N = 1$