EAS 342 Atmospheric Dynamics
Problem Set #6
Solution Notes

1. All the action is happening in the $y$-$z$ plane in this problem, since the zonal velocity is zero. Note also that this problem uses $p$ as the vertical coordinate, so the divergence must be written out using $p$ as the vertical coordinate. From the continuity equation, with $u=0$, 
\[ \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0. \]
You can either integrate this over both $y$ and $p$, or you can just write the derivatives as differences, since the velocities are specified to be constants on the entrance and exit surfaces. This gives 
\[ \frac{\partial \omega}{\partial p} = -\frac{\partial v}{\partial y} \Rightarrow \omega(900\text{hPa}) - \omega(800\text{hPa}) = \left[ \frac{(8\text{m/s} - 13\text{m/s})}{222\text{km}} \right] (100\text{hPa}) \]
\[ \omega(800\text{hPa}) = \omega(900\text{hPa}) + \left[ \frac{(8\text{m/s} - 13\text{m/s})}{222\text{km}} \right] (100\text{hPa}) = -1.25 \times 10^{-3}\text{hPa/s} \]
The air is rising at 800 hPa.

2. Problem 3.1 in Holton.

\[ V_g \]
\[ \theta \]
\[ V_a \]
\[ V \]

We are given that the angle $\alpha$ is 60 degrees, that $V_g$ is eastward with magnitude 225 m s$^{-1}$ and that the magnitude of $V_a$ is 200 m s$^{-1}$. From geostrophy, we can find $\partial \Phi / \partial x$ if we can calculate the meridional wind speed $(\partial \Phi / \partial x = f \nu)$. The diagram shows that the meridional wind speed must be equal and opposite to the northward air speed. Thus $\nu = -100$ m s$^{-1}$. Using $\Phi = gz$ and substituting into the geostrophic relationship gives 
\[ \frac{\partial z}{\partial x} = \frac{fv}{g} = -1 \frac{\text{m}}{\text{km}}. \]
3. Problem 3.2 in Holton. The trick is to remember that $\nabla \Phi$ is at right angles to $\mathbf{V}_g$ and to the right of $\mathbf{V}_g$. We are told that the wind velocity $\mathbf{V}$ is at angle $a = 30$ degrees to the right of $\mathbf{V}_g$. Thus we can construct the picture:

![Pressure Gradient Diagram]

Now use

$$|\nabla \Phi| = |f \mathbf{V}_g|,$$

and

$$\frac{\partial \Phi}{\partial s} = |\nabla \Phi| \sin a$$

Putting it all together and using Equation (3.9) from Holton gives

$$\frac{DV}{Dt} = -f \mathbf{V}_g \sin a = -10^{-3} \text{ m s}^{-2}$$

4. Problem 3.3 in Holton. Within a tornado it is reasonable to assume that the pressure gradient is balanced by the centrifugal acceleration. Rather than introducing natural coordinates it is simplest here to use the radius $r$ as the coordinate. We are given that the angular velocity $\omega$ is uniform through the core of the tornado. The tangential velocity $v$ is given by $v = \omega r$. The pressure gradient is given by

$$\frac{\partial p}{\partial r} = \rho \frac{v^2}{r}$$

Use the equation of state $p = \rho RT$ to evaluate $\rho$ in terms of the pressure and temperature, substitute for $v$ in terms of $\omega$, and one finds

$$\frac{dp}{p} = \frac{\omega^2}{RT} r \, dr.$$ 

This can be integrated from the center ($r = 0$) to $r_0$, to give

$$\ln \left[ \frac{p(r_0)}{p(0)} \right] = \frac{\omega^2}{RT} \frac{r_0^2}{2}.$$ 

Solving for $p(0)$ gives

$$p(0) = p(r_0) \exp \left( -\frac{\omega^2}{RT} \frac{r_0^2}{2} \right).$$

We are given the velocity at the outer edge $\omega r_0 = 100 \text{ m s}^{-1}$. Numerically we find $940 \text{ hPa}$ for the central pressure.
5. Problem 3.9 in Holton. Regular and anomalous highs both have $R < 0$. Both are clockwise circulations about high pressure centers. Therefore $\partial \Phi / \partial n < 0$ for both, because $n$ points outward towards lower pressure. The difference between the two cases is the ± in front of the square root in the solution of the quadratic equation for $V$, Equation (3.15) in the text. The question asks about the solutions in the limit of small $\partial \Phi / \partial n$. The procedure is to simplify the square root in this limiting case. The steps are:

\[
V = \frac{-fR}{2} \pm \left( \frac{f^2 R^2}{4} - R \frac{\partial \Phi}{\partial n} \right)^{1/2}
\]

\[
= \frac{-fR}{2} \pm \frac{|R|}{2} \left( 1 - \frac{4}{f^2 R^2} R \frac{\partial \Phi}{\partial n} \right)^{1/2}
\]

\[
= \frac{-fR}{2} \pm \frac{|R|}{2} \left( 1 - \frac{2}{f^2 R^2} R \frac{\partial \Phi}{\partial n} \right)
\]

\[
\approx f |R|, \quad - \frac{1}{f} \frac{\partial \Phi}{\partial n}.
\]

The last step retains only the largest term in the limit when $\partial \Phi / \partial n$ becomes very small. The first case uses the + sign and represents an inertial oscillation. The second case uses the – sign and represents geostrophic balance. Since $\partial \Phi / \partial n < 0$, $V$ is positive in both cases, so they are both physically possible.