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An approximate Riemann solver for relativistic magnetohydrodynamics

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ABSTRACT

A Godunov-type scheme for relativistic magnetohydrodynamic (MHD) equations is developed. We consider the Maxwell equations and dynamic equations for a gas with perfect conductivity in hyperbolic form as was suggested by van Putten. To calculate the fluxes of conservative variables through cells' interfaces we suggest an algorithm for the solution of the linearized Riemann problem. 'Primitive' variables are calculated by solving a non-linear system using the Newton method.

Key words: MHD – relativity – shock waves – methods: numerical.

1 INTRODUCTION

There are many astrophysical phenomena that should be described by relativistic magnetohydrodynamics (MHD): γ-ray bursts (Mészáros & Rees 1992; Rees & Mészáros 1994); relativistic jets from collapsars (Mirabel & Rodrígeez 1994, 1996; Levinson & Blandford 1996; Bogovalov 1997); young stellar objects (Appenzeller & Mundt 1989; Camenzind 1990; Montmerle et al. 1993; Fendt, Camenzind & Appl 1995); AGN (Begelman, Blandford & Rees 1984; Blandford, Netzer & Woltjer 1991; Dey & van Breugel 1994; Bhacall et al. 1995; Biretta, Zhou & Owen 1995; Boissé et al. 1998); and accretion on black holes (BH) (Khanna & Camenzind 1996). In particular, in the relativistic jets in AGN the Lorentz factor is considered to be as large as 10 and the magnetic field plays a significant role in this phenomenon (Begelman et al. 1984).

These flows can be described by relativistic MHD, considered in detail by Lichnerowicz (1967) and Anile (1989). This model is qualitatively similar to the model for non-relativistic MHD (assuming perfect conductivity). As well as for the non-relativistic case, the system of equations has three types of sound waves: alfvén-like and two magnetosonic (slow and fast) with the same relations between the wave speeds. Nevertheless there are some differences dealing with various invariant (with respect to the Lorentz transformation) forms of representation of the relativistic MHD equations. Depending on the form of this representation some types of virtual waves may arise: pseudo-contact and pseudo-alfvén-like waves (Komissarov 1999) or waves propagating with the speed of light (van Putten 1991).

The equations of relativistic MHD were investigated numerically by van Putten (1991, 1993, 1998), Komissarov (1999) and Balsara (2001). The first simulations of relativistic MHD were conducted by Dubal & Pantano (1993), who considered the steady-state case and used the method of characteristics, and van Putten (1994, 1996), where time-dependent relativistic MHD jets were calculated using the hyperbolic formulation by van Putten (1991, 1993, 1995). Later, astrophysical relativistic MHD flows were simulated by Bogovalov (1997) and Gómez (2001), see also a review by Martí & Müller (1999).

The present paper is mainly devoted to the construction of a Godunov-type scheme for Eulerian relativistic MHD, so this work is similar to the work by Komissarov (1999). The keypoint for the construction of the Godunov-type scheme is the calculation of fluxes of conservative variables at cells' interfaces. In the original paper by S. K. Godunov (Godunov 1959), these were calculated by means of an exact solution of the Riemann problem. Even in the case of gas dynamics, an iterative process is required and the calculation is very time-consuming. The situation becomes much more complicated for the MHD equations (Ryu & Jones 1995) so its inclusion in the computing algorithm is not expedient.

This is why a number of algorithms involving an approximate solution of the Riemann problem were suggested both for gas dynamics (Osher & Solomon 1982; Roe 1986) and MHD (Brio & Wu 1988).

The most popular approach for the approximate solution of the Riemann problem is perhaps that introduced by Roe (1986), who considered a linearized approximation, the effective values of the gasdynamical variables, wave speeds and right nullvectors being determined to obtain approximate solutions for the mass, momentum and energy of matter equal to their exact-solution values. The matrix of the linearized system has the same form as the Jacobian of the original system when substituting their effective values for the gas-dynamical variables, which have to be calculated in a special way.

Basically, we can apply this procedure to the MHD equations, but generally speaking this matrix will result from the substitution of any effective values into the matrix of the linearized system

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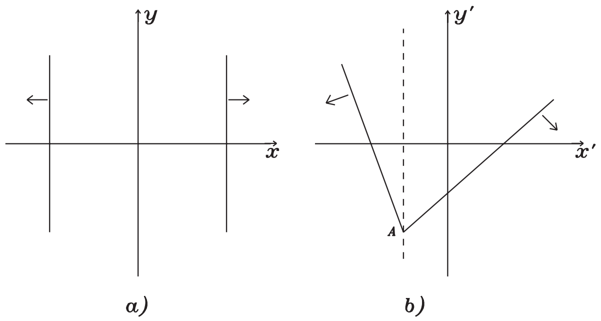

Figure 1. Wavefronts for the Riemann problem in the laboratory (a) and the moving (b) coordinate frames. In the laboratory frame, the fronts are parallel to the yz plane. In the moving frame, the decay has already has taken place above the point A, but not below it. The axes of the moving frame are oriented so that the decay proceeds along the $y'z'$ plane.

(Brio & Wu 1988). On the other hand, it was shown by Ryu & Jones (1995) that the way in which the effective values are calculated does not crucially influence the accuracy of the scheme.

Taking into account the above-mentioned considerations the main problem is to construct an algorithm for approximate solution of the Riemann problem for the relativistic MHD equations. Here the term 'approximate solution' is considered to be a solution of the linearized Riemann problem with respect to some background defined by effective values of gasdynamical variables. We assume (without asserting that our point of view is solely valid) that the choice of these effective values is not critical and can be made by any reasoning.

Let us consider the flow resulting from a decay of the discontinuity between two uniform states. At the initial moment of time these states are divided by the $x = 0$ plane in some coordinate frame that will be referred to hereinafter as the laboratory frame. Though in the laboratory frame the flow is one dimensional, this property does not persist when we transfer to another coordinate frame, since simultaneity is not absolute in relativistic mechanics. In other words, decay of the discontinuity occurs simultaneously at all points of the $x = 0$ plane in the laboratory frame but it does not take place in any other frame. Fig. 1 presents schematically the layout of wavefronts after decay of the discontinuity in the laboratory reference frame (Fig. 1a) and in some moving frame (Fig. 1b). In the laboratory frame (t, x, y, z) the fronts are parallel to the yz-plane. In a frame (t', x', y', z') moving with a constant speed the decay has already taken place above the point A, but not below it. The axes of the moving frame are oriented so that the decay occurs along the $y'z'$-plane. It is seen from Fig. 1(b) that the wavefronts are wedge shaped in the moving frame.

When building a Godunov-type scheme we should consider the system of conservation laws of hyperbolic type. In this case we can use the properties of characteristics of this system and approximate the fluxes of conservative variables in a fashion that provides stability of the explicit scheme for sufficiently small time-steps.[1] Basically, we could solve the Riemann problem staying in the laboratory frame and deal with a one-dimensional problem for a hyperbolic system consisting of seven equations for seven variables: three components of the velocity vector, two components of the magnetic field and two thermodynamical variables (for instance, pressure and entropy). However, formulae that arise in this case are enormous. Therefore this problem is usually considered in a covariant form (van Putten 1993; Komissarov 1999). This approach produces reasonably complex expressions for the wave amplitudes as well as other parameters but we have to pay the price of the multi-dimensionness of the Riemann problem since the laboratory frame is not especially chosen.

In the framework of this approach the components of four-vectors of velocity and magnetic field as well as two thermodynamic functions play the role of 'primitive' variables. Nevertheless these are not independent, but are connected by some constraints (see Section 2). This circumstance complicates both a theoretical analysis and the construction of numerical algorithms. To avoid these difficulties one can treat, following van Putten (1991, 1993), all 10 variables as independent but transform/extend the original system of equations so that the constraints are satisfied automatically (if they are fulfilled on the initial manifold, i.e. for $t = 0$ in the laboratory system). In some sense the situation is similar to the problem of maintaining div$\boldsymbol{H} = 0$ in non-relativistic MHD.

The system of equations of ideal relativistic MHD in covariant form contains four energy–momentum equations, four field equations and the continuity equation. To equalize the number of equations to the number of variables, the additional tenth equation is needed: the entropy equation (Anile & Pennisi 1987; Anile 1989; van Putten 1991), or the conservation equation for one of the constraints (van Putten 1993). Note that the resulting system of equations is, generally speaking, not of hyperbolic type. To convert it to a hyperbolic system some additional terms are added, these being formally equal to zero owing to constraints. Depending on the form of the additional terms, some types of virtual (physically meaningless) waves may arise: pseudo-contact and pseudo-alfvénic waves (Anile & Pennisi 1987; Komissarov 1999) or waves propagating with the speed of light (van Putten 1991). Of course, velocities and right nullvectors for physically-admissible waves are the same for all approaches and do not depend on the method of converting the system to a hyperbolic one.

It is worth stressing again that additional equations and additional terms (equal to zero for the physically-admissible solutions) are introduced to obtain the hyperbolic system in variables considered constraints free. In this work we adopt the equation of entropy conservation as an additional equation similar to the original work by van Putten (1991). We believe that closure of the system by the entropy conservation equation is reasonable since it permits us to include the calculation of the entropy flux in the solution of the Riemann problem, this flux being of importance for simulation of flows without shock waves. On the other hand, this approach permits us to simulate flows with shocks as well, since the entropy conservation equation is used only to obtain the Jacobian and to build an approximate Riemann solver. In Section 7 we will consider the system including the energy equation, which allows us to treat the entropy production at the shocks correctly.

In the standard approach to building the approximate Riemann solvers and calculating fluxes on the cells' interfaces, there exist some free parameters – namely the effective values of gasdynamical variables on the cells' interfaces. These effective values give the background for the linearized Riemann problem and, in particular, define the velocities of propagating waves and strengths of jumps. As an ideal solution, the effective values should be chosen so that the amplitudes of all physically non-admissible waves (formally calculated according to the formulae in Section 4) will go to zero. Unfortunately, as yet we cannot suggest an algorithm for calculating effective values that meets this condition. So in our scheme we make the amplitudes of virtual waves equal to zero 'by hand'.

[1] Defined by the value of Courant number.

Similar to others methods based on the solution of the Riemann problem in the linear (acoustic) approximation, our approach appears to admit the existence of unphysical jumps – namely rarefaction shock waves – but we did not investigate this in detail. The occurrence of these unphysical jumps is connected with the difference between the exact (non-linear) and approximate (linear) solutions of the Riemann problem: the latter is a set of jumps including those that should be centred rarefaction waves in the exact solution. These unphysical waves are characterized by decreasing entropy, and this circumstance can result in the occurrence of rarefaction shocks. The problem is not specific to relativistic MHD equations, but can also arise in the Roe-based approximation of gasdynamical equations (Einfeldt 1988). Solution methods for this problem are known (Einfeldt 1988), but they are beyond the scope of this paper.

Similar to the non-relativistic MHD case (Brio & Wu 1988), exploiting the approximate Riemann solver for relativistic MHD can result in the occurrence of degeneracies. We present the solution of the Riemann problem in a special form suitable for treatment of degeneracies in cases when (i) the component of the front perpendicular to the magnetic field vanishes; (ii) the component of the front parallel to the magnetic field vanishes. When dealing with relativistic MHD, the concepts of parallel and perpendicular components should be specified explicitly since these depend on the reference frame. We will consider this problem in Section 6. In this respect, our work is similar to that of Brio & Wu (1988) and is a sequel to the work of Komissarov (1999).

The paper is organized as follows: Section 2 presents the equations of relativistic MHD, in Section 3 we describe some relations on characteristic manifolds, Section 4 presents the solution of the Riemann problem in relativistic MHD, in Section 5 we analyse wave speeds, in Section 6 we consider the problem of degeneracies, in Section 7 the Godunov-type scheme is constructed, Section 8 describes the algorithm for calculation of ‘primitive’ variables, and Section 9 is devoted to numerical tests.

2 EQUATIONS OF RELATIVISTIC MHD

Within the framework of relativistic mechanics and electrodynamics we deal with a Minkowski space-time continuum equipped with the pseudo-Euclidian metric

$$g^{ik} = \mathrm{diag}(-1, 1, 1, 1).$$

Points of this continuum are labelled by coordinates $x^i = (x^0, x^1, x^2, x^3) = (t, x, y, z)$. An index notation is used for the space-time coordinates and tensor components throughout the paper except in some cases where the usual notation t, x, y, z is used (where it does not cause misunderstanding).

Kinematics of the fluid particles can be described by a four-velocity $u^i = (\gamma, \gamma\boldsymbol{v})$, where $\boldsymbol{v}$ is the velocity of matter and $\gamma = (1 - \boldsymbol{v}^2)^{-1/2}$ is the Lorentz factor. By virtue of our choice of the metric signature, $u^2 = u_i u^i = -1$.

For the description of magnetic field, let us introduce a magnetic field four-vector that, in the reference frame, is comoving with the plasma velocity (fluid frame) and has the form $h^{i'} = (0, \boldsymbol{H}'/\sqrt{4\pi})$, where the dash symbol denotes values in the fluid coordinate frame. A scalar product of four-vectors does not depend on the coordinate frame, so we have $h'_i h^{i'} = h_i h^i = \boldsymbol{H}'^2/4\pi$. The vectors u^i and h^i are orthogonal: $u_i h^i = 0$ (since they are orthogonal in the fluid frame: $u_{i'} h^{i'} = 0$). In terms of this field the first couple of Maxwell equations reads:

$$\frac{\partial}{\partial x^k}(h^i u^k - h^k u^i) = 0. \tag{1}$$

The second couple of Maxwell equations is not considered in the perfect MHD; the continuity and energy–momentum equations read:

$$\frac{\partial \rho u^k}{\partial x^k} = 0, \tag{2}$$

$$\frac{\partial (T_m^{ik} + T_f^{ik})}{\partial x^k} = 0. \tag{3}$$

Here $\rho = m_p n$; n is the number of baryons in the unit of space (in the fluid frame); m_p is rest mass of the proton; T_m^{ik}, T_f^{ik} are the energy–momentum tensors for matter and field, respectively, tensor T_m^{ik} being (Landau & Lifshitz 1959)

$$T_m^{ik} = w u^i u^k + p g^{ik},$$

where w is the enthalpy density and p is the pressure. Tensor T_f^{ik} can be expressed as (Lichnerowicz 1967; Anile 1989)

$$T_f^{ik} = h^2 u^i u^k + \frac{h^2}{2} g^{ik} - h^i h^k.$$

Using (2), (3) we can obtain the expression for the conservation of entropy:

$$\frac{\partial \rho s u^k}{\partial x^k} = 0, \tag{4}$$

where s is the specific entropy.

The system (1–3) contains 9 equation for 10 unknown variables that will be referred hereinafter as ‘primitive’ variables u^i, h^i, p, s. These are not independent and should satisfy constraints $u_k u^k = -1$, $u_k h^k = 0$. To avoid specified constraints and retain the covariant form of the equation, the following procedure was suggested by van Putten (1993):

(i) all ten variables u^i, h^i, p, s are considered to be independent;
(ii) a new equation is added to the system (1–3).

In contrast to van Putten (1993, 1998) but in a similar way to that adopted by van Putten (1991), we adopt the law of entropy conservation (4) as an additional equation. The constraints $u_k u^k = -1$, $u_k h^k = 0$ are superimposed at some initial manifold.

It is worth stressing here that system (1–4) is not hyperbolic; more strictly, we can say that any space-like plane is a characteristic surface. Actually, Maxwell's equations (1) include a relation div$\boldsymbol{H} = 0$ which contains a derivation with respect to space variables only, i.e. along the three-dimensional t = constant plane (in a corresponding reference frame). This means that the t = constant plane is the characteristic surface. By choosing an appropriate reference frame, we can convert any space-like plane to the form t = constant (in an adopted frame), so that it will be characteristic. To make the system (1–4) the hyperbolic one, van Putten (1993) suggested modifying Maxwell's equations by introducing an additional term $u_j h^j g^{ik} = 0$ under the derivative.

Finally, the extended system of relativistic MHD reads (van Putten 1998):

$$\left.\begin{aligned} &\frac{\partial T^{ik}}{\partial x^k} = 0, \quad \frac{\partial G^{ik}}{\partial x^k} = 0, \\ &\frac{\partial \rho u^k}{\partial x^k} = 0, \quad \frac{\partial \rho s u^k}{\partial x^k} = 0, \end{aligned}\right\} \tag{5}$$

where

$$T^{ik} = T^{ik}_m + T^{ik}_f$$

$$= (w + h^2)u^i u^k + \left(p + \frac{h^2}{2}\right)g^{ik} - h^i h^k,$$

$$G^{ik} = h^i u^k - h^k u^i + u_j h^j g^{ik}.$$

Note that values G^{10}, G^{20}, G^{30} are components of the three-vector $\boldsymbol{H}/\sqrt{4\pi}$.

3 RELATIONS ON CHARACTERISTIC MANIFOLDS

The main idea of the approximate solution of the Riemann problem is to solve it in an acoustic approximation. We assume that the values of jumps at the initial discontinuity, as well as those that arise after the decay of this discontinuity, are small enough. In this case the wave speeds (which are equal to appropriate sound speeds with respect to some effective state) and relations on discontinuities can be linearized with respect to this state. Therefore, it is first necessary to investigate the propagation of small-strength discontinuities along a uniform background.

In a similar way to that adopted by Anile (1989), let us introduce a 10-component 'vector' $\mathcal{U} = \{u^i, h^i, p, s\}$ and corresponding 'fluxes' $\mathcal{F}^k = \{T^{ik}, G^{ik}, \rho u^k, \rho s u^k\}$. In terms of these variables, equations (5) read

$$\frac{\partial \mathcal{F}^k}{\partial x^k} = 0.$$

Let Σ be a surface of the discontinuity in Minkowski space and ν_k be a four-vector orthogonal to this surface. Then the linearized Hugoniot relations on this surface can be expressed as follows:

$$\nu_k[\mathcal{F}^k] = \nu_k \mathcal{A}^k[\mathcal{U}] = 0, \tag{6}$$

where $[\ldots]$ is a jump of the variable and $\mathcal{A}^k = \partial\mathcal{F}^k/\partial\mathcal{U}$. It is seen from equation (6) that $[\mathcal{U}]$ is a right nullvector of $\nu_k\mathcal{A}^k$.

Note that for the original Maxwell equations (1), relations on the surface of the discontinuity (of small strength) read

$$\nu_k\{u^k[h^i] + h^i[u^k] - u^i[h^k] - h^k[u^i]\} = 0$$

and for any ν_k there exists their linear combination equal to zero, namely $\nu_i\nu_k\{\ldots\} \equiv 0$, since $\{\ldots\}$ is antisymmetric with respect to indices i, k. This means that any vector ν_k would be a characteristic vector of matrices $\mathcal{A}^k$ when the latter are calculated from the original Maxwell equations, since $\det(\nu_k\mathcal{A}^k) = 0$.

Let us treat the system as a perfect gas with adiabatic index Γ, then

$$w = \rho + \frac{\Gamma p}{\Gamma - 1},$$

$$p = K(s)\rho^\Gamma,$$

and the gas sound speed in this case is $c^2 = \Gamma p/w$ (Landau & Lifshitz 1959).

Considering $[u^i]$, $[h^i]$, $[p]$, $[s]$ as unknown variables in equation (6) and equating the determinant to zero we can conclude that the components of four-vector ν_i should satisfy one of the following relations:

$$\left.\begin{array}{ll}
\text{(i)} & \omega^2 = 0;\\
\text{(ii)} & \omega^2 = \dfrac{\eta^2}{w + h^2};\\
\text{(iii)} & (1 - c^2)\omega^4 - \left(c^2 + \dfrac{h^2}{w}\right)\nu^2\omega^2 + \dfrac{c^2}{w}\eta^2\nu^2 = 0;\\
\text{(iv)} & \nu^2 = 0;
\end{array}\right\} \tag{7}$$

where $\omega = \nu_k u^k$, $\eta = \nu_k h^k$, $\nu^2 = \nu_k\nu^k$. Case (i) corresponds to entropic waves, case (ii) to alfvénic waves, case (iii) to magnetosonic (slow and fast) waves, and case (iv) to waves propagating with the speed of light.

For each of these waves there exists a right nullvector, but some of them are not physically admissible – recall that the jumps of u^i, h^i, p, s on the discontinuity are proportional to the respective components of the right nullvector. However, these jumps are not arbitrary values since they should satisfy relations

$$u_k[u^k] = 0, \quad h_k[u^k] + u_k[h^k] = 0 \tag{8}$$

resulting from the constraints $u_k u^k = -1$, $u_k h^k = 0$. Furthermore, in the laboratory frame we have $H_x =$ constant To proceed with this condition in covariant form, let us introduce four-vectors ξ^k, ζ^k with components $\xi^k = (1, 0, 0, 0)$, $\zeta^k = (0, 1, 0, 0)$ in the laboratory frame. Then the condition $H_x =$ constant can be written as $\chi_{ik}h^i u^k =$ constant, where $\chi_{ik} = \xi_i\zeta_k - \xi_k\zeta_i$. The jumps on physically-admissible waves should satisfy

$$\chi_{ik}(h^i[u^k] - u^i[h^k]) = 0. \tag{9}$$

To make the notation of the relations on characteristic surfaces more compact it is convenient to introduce an orthonormal basis (one for each characteristic surface) in Minkowski space (van Putten 1998). Let us define

$$n^i = \nu^i + \omega u^i,$$

then $n_i u^i = 0$. The geometry of the vector n^i is as follows. In the fluid reference frame, $n^{i'} = (0, \boldsymbol{n}')$ – where $\boldsymbol{n}'$ is a unit vector orthogonal to the intersection of surface Σ with the $t' =$ constant plane (i.e. with the surface of the front at some moment of time in the fluid frame). We assume that the vectors n^i and ν^i are normalized as $n^2 = n_i n^i = 1$, then $\nu^2 = 1 - \omega^2$. Let us also define

$$f^i = h^i - \eta n^i.$$

It is seen that

$$u_i f^i = 0,$$

$$n_i f^i = 0,$$

$$h^2 = f^2 + \eta^2.$$

Let us also introduce $f^i = f t^i$ (if $f = 0$ then t^i is undefined, see Section 6.1). The vectors u^i, n^i, t^i give an orthonormal system. To obtain a basis, it is necessary to complete this system by a vector $e^i = \varepsilon^{ijkl}u_j n_k t_l$ that is orthogonal to u^i, n^i, t^i and $e^2 = e_i e^i = 1$ (ε^{ijkl} is the Levi–Civita alternating tensor). Let us investigate various types of waves supposing that $\eta \neq 0$, $c \neq 0$.

(i) Consider the case $\omega = \nu_k u^k = 0$ (it is marked by index E) corresponding to standing waves in the fluid frame. The right nullvectors of $\nu_k\mathcal{A}^k$ are

$$\mathcal{R}_E = \begin{Bmatrix} 0^i \\ 0^i \\ 0 \\ 1 \end{Bmatrix}, \qquad \mathcal{R}'_E = \begin{Bmatrix} u^i \\ h^i - 2\eta n^i \\ -\dfrac{\rho h^2}{wc^2} \\ 0 \end{Bmatrix}.$$

The nullvector $\mathcal{R}'_E$ does not satisfy the conditions (8) and corresponding wave should be excluded. On the wave corresponding to the nullvector $\mathcal{R}_E$ only the entropy has a jump so it is the entropic wave.

(ii) For the alfvénic waves (index A)

$$\omega = \pm \frac{\eta}{\sqrt{w+h^2}},$$

and the right nullvectors are

$$\mathcal{R}_{\pm A} = \begin{Bmatrix} \pm e^i \\ \sqrt{w+h^2} e^i \\ 0 \\ 0 \end{Bmatrix}.$$

(iii) For the magnetosonic waves (index M) the characteristic equations (7, iii) can be rewritten as

$$\omega^4 - (c^2 + c_A^2 + (1-c^2) c_A^{\parallel 2}) \quad \omega^2 + c^2 c_A^2 = 0, \tag{10}$$

where $c_A^{\perp} = \eta/\sqrt{w+h^2}$, $c_A^{\parallel} = f/\sqrt{w+h^2}$ – alfvénic velocities in the normal and tangential to the front directions respectively and $c_A = |c_A^{\perp}|$. The roots of this equation $\pm c_{\mathrm{F}}$ and $\pm c_{\mathrm{S}}$ are the velocities of fast and slow magnetosonic waves in the normal direction in the fluid frame.

Let ω be a root of characteristic equation, and ω_* is another root such that $\omega\omega_* = cc_A$, then corresponding right nullvector can be written as

$$\mathcal{R}_M = \begin{Bmatrix} -\alpha\omega n^i + \beta\sigma\omega_*\sqrt{1-c^2}\, t^i \\ c\sqrt{w+h^2}\left[-\alpha\sigma\omega_* u^i + \beta\dfrac{1-\omega_*^2}{\sqrt{1-c^2}} t^i\right] \\ \alpha\Gamma p \\ 0 \end{Bmatrix}.$$

Here $\sigma = \mathrm{sign}(\eta)$ and the index M runs four values $\pm F$, $\pm S$ which correspond to the fast and slow magnetosonic waves. For the fast magnetosonic waves we have:

$$\alpha = \sqrt{\frac{c^2 - c_{\mathrm{S}}^2}{c_{\mathrm{F}}^2 - c_{\mathrm{S}}^2}}, \quad \beta = \sqrt{\frac{c_{\mathrm{F}}^2 - c^2}{c_{\mathrm{F}}^2 - c_{\mathrm{S}}^2}},$$

$$\omega^2 = c_{\mathrm{F}}^2, \quad \omega_*^2 = c_{\mathrm{S}}^2.$$

For the slow magnetosonic waves we have:

$$\alpha = -\sqrt{\frac{c_{\mathrm{F}}^2 - c^2}{c_{\mathrm{F}}^2 - c_{\mathrm{S}}^2}}, \quad \beta = \sqrt{\frac{c^2 - c_{\mathrm{S}}^2}{c_{\mathrm{F}}^2 - c_{\mathrm{S}}^2}},$$

$$\omega^2 = c_{\mathrm{S}}^2, \quad \omega_*^2 = c_{\mathrm{F}}^2.$$

For the waves propagating to the right $\omega < 0$, $\omega_* < 0$, and for ones propagating to the left $\omega > 0$, $\omega_* > 0$.(iv) For the waves propagating with the speed of light the nullvectors $\mathcal{R}_C, \mathcal{R}_C'$ don't satisfy the condition (9) and these waves should be excluded.

Note that the four-vectors $[u^i]_a, [h^i]_a$ (here the index a runs through values E, $\pm A$, $\pm S$ and $\pm F$) satisfy condition

$$\chi_{ik}(u^i [h^k]_a + [u^i]_a h^k) = 0,$$

i.e. the component of the magnetic field normal to the front (in the laboratory frame) $H_x = \sqrt{4\pi}\chi_{ik} h^i u^k$ is continuous on these waves.

The 'ten-vectors' $\{\mathcal{R}_E, \mathcal{R}_{\pm A}, \mathcal{R}_{\pm F}, \mathcal{R}_{\pm S}\}$ form the basis of a seven-dimensional subspace that is orthogonal to the following three 'ten-vectors' of conjugate space: $\{u_i, 0_i, 0, 0\}$, $\{h_i, u_i, 0, 0\}$, $\{0_i, n_i, 0, 0\}$. Let us construct a basis $\{\mathcal{P}_E, \mathcal{P}_{\pm A}, \mathcal{P}_{\pm F}, \mathcal{P}_{\pm S}\}$ that is adjoint to the basis $\{\mathcal{R}_E, \mathcal{R}_{\pm A}, \mathcal{R}_{\pm F}, \mathcal{R}_{\pm S}\}$ in the sense

$$\langle \mathcal{P}_a, \mathcal{R}_b \rangle = \delta_{ab}, \quad (a, b = E, \pm A, \pm F, \pm S).$$

In the laboratory reference frame the equations of the characteristic surfaces are

$$\phi(x^k) = x - \lambda t = \text{constant}$$

and the characteristic vectors are

$$\nu_k = \partial\phi/\partial x^k = (-\lambda, 1, 0, 0) = \lambda\xi_k + \zeta_k,$$

so the matrices $\nu_k \mathcal{A}^k$ can be expressed as

$$\nu_k \mathcal{A}^k = \lambda\xi_k \mathcal{A}^k + \zeta_k \mathcal{A}^k.$$

Let $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{L}$, $\mathcal{L}'$ be the right and left nullvectors corresponding to different eigenvalues λ and λ', then

$$\lambda\xi_k \mathcal{A}^k \mathcal{R} + \zeta_k \mathcal{A}^k \mathcal{R} = 0,$$

$$\lambda' \mathcal{L}' \xi_k \mathcal{A}^k + \mathcal{L}' \zeta_k \mathcal{A}^k = 0.$$

Combine these two to obtain

$$(\lambda' - \lambda)\mathcal{L}' \xi_k \mathcal{A}^k \mathcal{R} = 0.$$

This implies that the adjoint basis is constituted by the 'vectors'

$$\mathcal{P}_E = \mathcal{L}_E \xi_k \mathcal{A}^k,$$

$$\mathcal{P}_{\pm A} = \mathcal{L}_{\pm A} \xi_k \mathcal{A}^k,$$

$$\mathcal{P}_M = \mathcal{L}_M \xi_k \mathcal{A}^k,$$

where $\mathcal{L}_E$, $\mathcal{L}_{\pm A}$, $\mathcal{L}_M$ are left nullvectors of the matrices $\nu_k \mathcal{A}^k$. The 'vectors' of the adjoint basis are as follows (Koldoba & Ustyugova 1999):

$$\mathcal{P}_E = \{0_i, 0_i, 0, 1\},$$

$$\mathcal{P}_{\pm A} = \begin{Bmatrix} (\pm\xi_k s^k - c_A^{\parallel}\xi_k t^k) e_i \\ \dfrac{e_i(\xi_k s^k \mp c_A^{\parallel}\xi_k t^k) \pm \xi_k e^k (c_A^{\perp} n_i + c_A^{\parallel} t_i)}{\sqrt{w+h^2}} \\ \pm\dfrac{e_k \xi^k}{w+h^2} \\ 0 \end{Bmatrix}^{\mathrm{T}},$$

$$\mathcal{P}_M = \begin{Bmatrix} \alpha\omega(\omega\xi_i - \xi_k u^k n_i) + \dfrac{\beta t_i(\sigma\omega_*\xi_k s^k - c_A^{\parallel} c\,\xi_k t^k)}{\sqrt{1-c^2}} \\ \dfrac{\beta c}{\sqrt{(1-c^2)(w+h^2)}}(\xi_k s^k t_i + \omega\xi_k t^k n_i) \\ \dfrac{1-\omega^2}{w(1-c^2)}\left(\alpha\xi_k s^k + \beta\sigma\sqrt{1-c^2}\,\omega_*\xi_k t^k\right) \\ 0 \end{Bmatrix}^{\mathrm{T}},$$

where $s^i = u^i - \omega n^i$ is a projection of the four-velocity onto the characteristic manifold. The non-zero scalar products are

$$\langle \mathcal{P}_E, \mathcal{R}_E \rangle = 1,$$

$$\langle \mathcal{P}_{\pm A}, \mathcal{R}_{\pm A} \rangle = 2(\xi_k s^k \mp c_A^{\parallel}\xi_k t^k),$$

$$\langle \mathcal{P}_M, \mathcal{R}_M \rangle = 2c^2\left(\xi_k s^k + \frac{\alpha\beta\sigma\omega_*(1-\omega^2)}{\sqrt{1-c^2}}\xi_k t^k\right).$$

Note that the vectors n^k, t^k, e^k, s^k and scalars c_F, c_S, c_A, α, β are different for various waves since the vectors ν^k are also different.

4 RIEMANN PROBLEM

Consider a Riemann problem for the system (5). Let the $x = 0$ plane divide two uniform states marked by L $(x < 0)$ and R $(x > 0)$ in the laboratory frame at the moment of time $t = 0$. In both states the variables satisfy $u_i u^i = -1$ and $u_i h^i = 0$. Besides, we should superimpose the condition of continuity of the normal component of the magnetic field $H_{x\mathrm{L}} = H_{x\mathrm{R}} = H_{x0}$ or in the invariant form $\chi_{ik} h^i u^k = H_{x0}/\sqrt{4\pi}$. The right nullvectors of physically-admissible waves should be orthogonal to the row 'vectors'

$$\left.\begin{array}{l}\{u_i, 0_i, 0, 0\},\\ \{h_i, u_i, 0, 0\},\\ \{\chi_{ik} h^k, -\chi_{ik} u^k, 0, 0\},\end{array}\right\} \qquad (11)$$

which implies continuity of the values $u_i u^i, u_i h^i$ and $\chi_{ik} h^i u^k$ on the wave fronts. It is easy to check that the 'vectors' $\mathcal{R}_a$ constructed above are orthogonal to the row vectors (11).

Decay of the initial discontinuity results in the propagation of waves, on every wave the variables $\mathcal{U} = \{u^i, h^i, p, s\}$ suffering a jump, the 'vector' of the jump being collinear to the right nullvector corresponding to this wave. Since the right nullvectors of all waves (both physically admissible and non-admissible) constitute a basis in the ten-dimensional space of 'vectors' $\mathcal{U}$, then the strengths of jumps $\Delta\mathcal{U} = \mathcal{U}_\mathrm{R} - \mathcal{U}_\mathrm{L}$ can be expressed as

$$\Delta\mathcal{U} = \sum_a \langle\mathcal{P}_a, \Delta\mathcal{U}\rangle\mathcal{R}_a,$$

where $\{\mathcal{P}_a\}$ is a basis adjoint to $\{\mathcal{R}_a\}$: $\langle\mathcal{P}_a, \mathcal{R}_b\rangle = \delta_{ab}$. Owing to the conditions that were superimposed on the left and right states we can assert that the 'vector' $\Delta\mathcal{U}$ belongs to the subspace spanned by the 'vectors' $\{\mathcal{R}_E, \mathcal{R}_{\pm A}, \mathcal{R}_{\pm S}, \mathcal{R}_{\pm F}\}$, i.e. the amplitudes of the pseudo-entropic wave and light waves are equal to zero.

To construct a Godunov-type scheme we should calculate the values of specific fluxes (fluxes per unit area) through the *YZ* plane in the laboratory frame. In this frame the solution of the Riemann problem (both exact and approximate) is one dimensional and self-similar so in the *YZ* plane all values are constant in time. When passing through the wavefront of type $a = \{E, \pm A, \pm S, \pm F\}$ in the direction $L \rightarrow R$ the primitive variables suffer a jump $[\mathcal{U}]_a = \langle\mathcal{P}_a, \Delta\mathcal{U}\rangle\mathcal{R}_a$. Consequently the fluxes $\zeta_k\mathcal{F}^k$ in the x-direction (of the laboratory reference frame) suffer a jump equal to $\zeta_k[\mathcal{F}^k]_a$. Since $\nu_k = \lambda_a\xi_k + \zeta_k$ we have

$$\zeta_k[\mathcal{F}^k]_a = \zeta_k\mathcal{A}^k[\mathcal{U}]_a = -\lambda_a\xi_k\mathcal{A}^k[\mathcal{U}]_a + \nu_k\mathcal{A}^k[\mathcal{U}]_a.$$

The second term is equal to zero, so we have

$$\zeta_k[\mathcal{F}^k]_a = -\lambda_a\xi_k\mathcal{A}^k[\mathcal{U}]_a = -\lambda_a\xi_k\mathcal{A}^k\mathcal{R}_a\langle\mathcal{P}_a, \Delta\mathcal{U}\rangle.$$

The values $\mathcal{A}^k\mathcal{R}_a$ are jumps of the fluxes $\mathcal{F}^k$ on the wave of type a having a unit amplitude $[\mathcal{U}]_a = \mathcal{R}_a$. Therefore the fluxes in the *YZ* plane can be approximately calculated as

$$\mathcal{F}^x = \frac{1}{2}(\zeta_k\mathcal{F}^k(-0) + \zeta_k\mathcal{F}^k(+0))$$
$$= \frac{1}{2}(\zeta_k\mathcal{F}^k_\mathrm{L} + \zeta_k\mathcal{F}^k_\mathrm{R}) + \frac{1}{2}\sum_a|\lambda_a|\mathcal{Q}_a\langle\mathcal{P}_a, \Delta\mathcal{U}\rangle, \qquad (12)$$

where $\mathcal{Q}_a = \xi_k\mathcal{A}^k\mathcal{R}_a$.

Finally the jumps of the primitive variables and fluxes read as follows.

(i) Entropic wave ($a = E$).

$$[u^i]_E = 0, \quad [h^i]_E = 0, \quad [p]_E = 0,$$

$$[T^{ik}]_E = [\rho]_E u^i u^k = \left(\frac{\partial\rho}{\partial s}\right)_p [s]_E u^i u^k = -\frac{\rho^{\Gamma+1}}{\Gamma p}u^i u^k,$$

$$[G^{ik}]_E = 0,$$

$$[\rho u^k]_E = [\rho]_E u^k = -\frac{\rho^{\Gamma+1}}{\Gamma p}u^k.$$

(ii) Alfvén waves ($a = \pm A$).

$$[s]_{\pm A} = 0, \quad [p]_{\pm A} = 0, \quad [h^2]_{\pm A} = 0,$$

$$[T^{ik}]_{\pm A} = (w + h^2)[\pm(e^i s^k + e^k s^i) - c_A^{\|}(t^i e^k + t^k e^i)],$$

$$[G^{ik}]_{\pm A} = \sqrt{w + h^2}[e^i s^k - e^k s^i \pm c_A^{\|}(t^i e^k - t^k e^i)],$$

$$[\rho u^k]_{\pm A} = \pm\rho e^k.$$

(iii) Magnetosonic waves ($a = M$).

$$[s]_M = 0,$$

$$[T^{ik}]_M = k_1(n^i u^k + n^k u^i) + k_2(t^i s^k + t^k s^i)$$
$$+ k_3\left(u^i u^k - t^i t^k + \frac{1}{2}g^{ik}\right)$$
$$+ \alpha\Gamma p\left(g^{ik} + \frac{1 + c^2}{c^2}u^i u^k\right),$$

$$[G^{ik}]_M = \beta c(1 - \omega_*^2)\sqrt{\frac{w + h^2}{1 - c^2}}(t^i s^k - t^k s^i),$$

$$[\rho u^k]_M = \rho\left(\alpha s^k + \beta\sigma\omega_*\sqrt{1 - c^2}t^k\right),$$

where

$$k_1 = -\alpha\omega(1 - \omega_*^2)(w + h^2),$$

$$k_2 = \frac{\beta\sigma\omega_*(1 - \omega_*^2)}{\sqrt{1 - c^2}}(w + h^2),$$

$$k_3 = \frac{2\beta c_A^{\|} c(1 - \omega_*^2)}{\sqrt{1 - c^2}}(w + h^2).$$

5 WAVE SPEEDS

Wave speeds in the laboratory frame are calculated as follows. Let λ is a wave speed in the laboratory frame. Then characteristic manifold in this frame is described by the equation $x^1 - \lambda x^0 =$ constant, and, consequently, $\nu_k = (-\lambda, 1, 0, 0)$. Let $\boldsymbol{v} = (v^1, v^2, v^3)$ is a velocity vector, and $u^k = (\gamma, \gamma\boldsymbol{v})$, $h^k = (h^0, \boldsymbol{h})$ are four-vectors of velocity and magnetic field. Since the four-vectors u^k and h^k are orthogonal we have $h^0 = (\boldsymbol{v}\cdot\boldsymbol{h})$. Furthermore,

$$\omega = \nu_k u^k = \gamma(v^1 - \lambda),$$

$$\nu^2 = 1 - \lambda^2,$$

$$\eta = \nu_k h^k = -\lambda h^0 + h^1 = -\lambda(\boldsymbol{v}\cdot\boldsymbol{h}) + h^1,$$

$$h^2 = h_i h^i = -h^{02} + \boldsymbol{h}^2 = \boldsymbol{h}^2 - (\boldsymbol{v}\cdot\boldsymbol{h})^2.$$

Combining these equations with (7) we get:

(i) Entropic wave: $\omega = 0$, so $\lambda_E = v^1 = v_x$.

(ii) Alfvén waves: $\omega = \pm\eta/\sqrt{w+h^2}$. Substitution for ω and η yields

$$\lambda_{\pm A} = \frac{u^1\sqrt{w+h^2} \mp h^1}{u^0\sqrt{w+h^2} \mp h^0}.$$

(iii) Magnetosonic waves: a substitution for $\omega^2, \eta^2, \nu^2, h^2$ in (7, iii) gives an equation of fourth order in λ. Its roots ordering descendant give the speeds of fast and slow waves propagating to the left and slow and fast waves propagating to the right in the laboratory frame. To calculate these speeds it is convenient to transfer in a reference frame moving with the velocity $V = v^1$ with respect to laboratory frame. The wave speed in the new frame is connected with λ by

$$\lambda = \frac{V + \mu}{1 + V\mu}. \tag{13}$$

Substituting equation (13) into part (iii) of equation (7) yields an equation for μ:

$$(1-c^2)\mu^4 - \left(\frac{h^2}{w} + c^2\right)\frac{\mu^2(1-\mu^2)}{\gamma^2(1-V^2)} + \frac{c^2}{w}\frac{(A+B\mu)^2(1-\mu^2)}{\gamma^4(1-V^2)^3} = 0, \tag{14}$$

where $A = h^1 - Vh^0 = H_x/\gamma\sqrt{4\pi}$, $B = Vh^1 - h^0$. If $A = 0$ then equation (14) has a twofold root $\mu = 0$. In this degenerate case $\eta = \nu_k h^k = 0$, $c_A = 0$, $c_S = 0$ (see Section 6.2).

6 DEGENERACIES

Degeneracies arise in the formulas for the 'vectors' $\mathcal{R}_a$ and $\mathcal{P}_a$. The source for appearance of the degeneracies is as follows: some variables involved in these expressions are defined ambiguously. For example, if $h^i = \eta n^i$ then the vector t^i (and, correspondingly, e^i) can be defined with some arbitrariness. This ambiguity is developed in the expressions for 'vectors' $\mathcal{R}_{\pm A}$, $\mathcal{R}_M$, $\mathcal{P}_{\pm A}$ and $\mathcal{P}_M$, which appeal to the vectors t^i, e^i. Similarly, in the case of $\eta = 0$ the variable $\sigma = \mathrm{sign}(\eta)$ is undefined, which results in an ambiguity in expressions for the 'vectors' $\mathcal{R}_M$, $\mathcal{P}_M$. Nevertheless, a combined expression

$$\sum_a |\lambda_a| \mathcal{Q}_a \langle \mathcal{P}_a, \Delta\mathcal{U}\rangle = \xi_k \mathcal{A}^k \sum_a |\lambda_a| \mathcal{R}_a \langle \mathcal{P}_a, \Delta\mathcal{U}\rangle$$

is always defined unambiguously and does not depend on the choice of the vector t^i (and, correspondingly, e^i) for the first case or on the choice of $\sigma = \pm 1$ for the second case.

6.1 Degeneracy 1: $h^i = \eta n^i$

Here the magnetic field only has a component normal to the front in the fluid reference frame: $h^{i'} = (0, \boldsymbol{H}'/\sqrt{4\pi})$, $n^{i'} = (0, \boldsymbol{n}')$, and $\boldsymbol{H}' \| \boldsymbol{n}'$ ($\boldsymbol{n}'$ is a unit vector normal to the front). In this case the vectors t^i and e^i are undefined, $c_A^{\|} = 0$ and the dispersional equation (10) for the magnetosonic waves reads:

$$\omega^4 - (c^2 + c_A^2)\omega^2 + c^2 c_A^2 = 0,$$

so the roots are $\omega_1^2 = c^2$, $\omega_2^2 = c_A^2$.

First, let us assume that $c_A^{\perp} = \eta/\sqrt{w+h^2} > 0$. The following mathematics depends on the relation between the speed of sound c and the alfvénic speed c_A: (i) if $c^2 < c_A^2$ then $\omega_{\pm S}^2 = c^2$ and $\omega_{\pm F}^2 = c_A^2$; (ii) if $c^2 > c_A^2$ then $\omega_{\pm S}^2 = c_A^2$ and $\omega_{\pm F}^2 = c^2$ (the case $c^2 = c_A^2$ will be considered elsewhere).

In case (i), the wave involved is the fast magnetosonic wave and its characteristic surface coincides with the characteristic surface of the alfvénic wave moving in the same direction, so $\omega^2 = c_F^2 = c_A^2$, $\omega_*^2 = c_S^2 = c^2$, $\alpha = 0$, $\beta = 1$.

We can prove that the combined expression

$$\mathcal{G} = \mathcal{R}_{\pm F}\langle \mathcal{P}_{\pm F}, \Delta\mathcal{U}\rangle + \mathcal{R}_{\pm A}\langle \mathcal{P}_{\pm A}, \Delta\mathcal{U}\rangle$$

does not depend on the choice of the vectors t^i, e^i. Indeed, we have

$$\langle \mathcal{P}_{\pm A}, \mathcal{R}_{\pm A}\rangle = 2\xi_k s^k, \qquad \langle \mathcal{P}_{\pm F}, \mathcal{R}_{\pm F}\rangle = 2c^2\xi_k s^k,$$

the vector s^k being the same for the fast magnetosonic and alfvénic waves considered that move in the same direction. The expressions for the right nullvectors and wave amplitudes now read:

$$\mathcal{R}_{\pm A} = \begin{Bmatrix} \pm e^i \\ \sqrt{w+h^2}e^i \\ 0 \\ 0 \end{Bmatrix},$$

$$\mathcal{R}_{\pm F} = c\sqrt{1-c^2}\begin{Bmatrix} \pm t^i \\ \sqrt{w+h^2}t^i \\ 0 \\ 0 \end{Bmatrix},$$

$$\langle \mathcal{P}_{\pm A}, \Delta\mathcal{U}\rangle = \pm e_k \Delta u^k + \frac{e_k \pm c_A^{\perp}\varepsilon n_k}{\sqrt{w+h^2}}\Delta h^k \pm \frac{\varepsilon}{w+h^2}\Delta p,$$

$$\langle \mathcal{P}_{\pm F}, \Delta\mathcal{U}\rangle = \frac{1}{c\sqrt{1-c^2}}\left(\pm t_k \Delta u^k + \frac{t_k \pm c_A^{\perp}\tau n_k}{\sqrt{w+h^2}}\Delta h^k \pm \frac{\tau}{w+h^2}\Delta p\right).$$

Here $\varepsilon = \xi_k e^k/\theta$, $\tau = \xi_k t^k/\theta$, $\theta = \xi_k s^k$.

Let us consider the first four components of the 'vector' $\mathcal{G}$; these components constitute a four-vector in Minkowski space. We have

$$g^i = e^i e_k \Delta u^k + t^i t_k \Delta u^k \pm e^i \frac{e_k \Delta h^k}{\sqrt{w+h^2}} \pm t^i \frac{t_k \Delta h^k}{\sqrt{w+h^2}} + e^i \frac{c_A^{\perp}\varepsilon n_k \Delta h^k}{\sqrt{w+h^2}} + t^i \frac{c_A^{\perp}\tau n_k \Delta h^k}{\sqrt{w+h^2}} + e^i \frac{\varepsilon\Delta p}{w+h^2} + t^i \frac{\tau \Delta p}{w+h^2} = (e^i e_k + t^i t_k) f^k,$$

where

$$f^k = \Delta u^k \pm \frac{\Delta h^k}{\sqrt{w+h^2}} + \frac{c_A^{\perp} n_j \Delta h^j}{\theta\sqrt{w+h^2}}\xi^k + \frac{\Delta p}{\theta\left(w+h^2\right)}\xi^k$$

does not depend on the choice of the vectors t^i and e^i.

Note that the vector $g^i = (e^i e_k + t^i t_k) f^k$ is a projection of the vector f^i on the plane constituted by the vectors e^i and t^i. Since the vector g^i can be expressed as

$$g^i = f^i + u^i u_k f^k - n^i n_k f^k,$$

this vector does not depend on the choice of unit vectors e^i and t^i if the vectors e^i, t^i, u^i and n^i constitute an orthonormal basis in Minkowski space.

The situation for the next four components of the 'vector' $\mathcal{G}$, which also constitute a four-vector, is the same. The last two component of the 'vector' $\mathcal{G}$ are the scalars, which are equal to zero.

Thus, the 'vector' $\mathcal{G}$ does not depend on the choice of unit vector t^i (nor on the choice of e^i) if this vector is orthogonal to u^i, n^i.

Consideration of the case (ii) is similar and it is easy to show that the combined expression

$$\mathcal{G} = \mathcal{R}_{\pm S}\langle \mathcal{P}_{\pm S}, \Delta\mathcal{U}\rangle + \mathcal{R}_{\pm A}\langle \mathcal{P}_{\pm A}, \Delta\mathcal{U}\rangle$$

does not depend on the choice of the vector t^i.

And, finally, for the case $c_A^{\perp} < 0$ the second term of the expressions for $\mathcal{G}$ should be taken as $\mathcal{R}_{\mp A}\langle \mathcal{P}_{\mp A}, \Delta\mathcal{U}\rangle$.

6.2 Degeneracy 2: $\eta = 0$

Now let us consider the degeneracy arising in the expressions for the magnetosonic right nullvectors $\mathcal{R}_M$ and wave amplitudes $\langle \mathcal{P}_M, \Delta\mathcal{U}\rangle$ when $\eta = \nu_k h^k = 0$ and the value of $\sigma = \mathrm{sign}(\eta)$ is undefined. Note that $\mathcal{R}_M$ and $\langle \mathcal{P}_M, \Delta\mathcal{U}\rangle$ involve the variable σ in a context $\sigma\omega_*$. Suppose that $\eta = 0$ for some magnetosonic wave (fast or slow), then the dispersional equations (10) become

$$\omega^4 + [c^2 + (1 - c^2)c_A^{\parallel 2}]\omega^2 = 0.$$

In the case of the fast magnetosonic wave we have $\omega_* = 0$ and the degeneracy is resolved. In the case of the slow magnetosonic wave we have $\omega = 0$ and the wave speed in the laboratory frame is $\lambda = v_x$, and, correspondingly, $\mu = 0$ (see equation 13). Being a root of equation (14), $\mu = 0$ implies that (i) $A = h^1 - Vh^0 = 0$, $H_x = 0$, and (ii) the root $\mu = 0$ is twofold.

Thus, characteristic manifolds for the slow magnetosonic waves propagating to the right and to the left coincide: $\lambda_{+S} = \lambda_{-S}$ and the vectors t^i, e^i, s^i coincide as well. Let us prove that a combined expression

$$\mathcal{G} = \mathcal{R}_{+S}\langle \mathcal{P}_{+S}, \Delta\mathcal{U}\rangle + \mathcal{R}_{-S}\langle \mathcal{P}_{-S}, \Delta\mathcal{U}\rangle$$

does not depend on the choice of $\sigma = \mathrm{sign}(\eta) = \pm 1$. Note that the expressions for $\mathcal{R}_{\pm S}$, $\langle \mathcal{P}_{\pm S}, \Delta\mathcal{U}\rangle$, $\langle \mathcal{P}_{\pm S}, \mathcal{R}_{\pm S}\rangle$ involve the variable σ linearly, i.e. in the form $X \pm \sigma Y$, where X and Y are expressions not including σ and not depending on the direction of wave propagation. It is seen that operations of the multiplication and division retain this form:

$$(X_1 \pm \sigma Y_1)(X_2 \pm \sigma Y_2) = X_3 \pm \sigma Y_3,$$

$$\frac{X_1 \pm \sigma Y_1}{X_2 \pm \sigma Y_2} = X_3 \pm \sigma Y_3.$$

Thus, the expression

$$\mathcal{G} = (\mathcal{A} + \sigma\mathcal{B}) + (\mathcal{A} - \sigma\mathcal{B}) = 2\mathcal{A}$$

does not include the undefined variable σ.

7 GODUNOV-TYPE SCHEME

We use the procedure described above for approximate solution of the Riemann problem to construct the Godunov-type scheme for relativistic MHD. In regular gridpoints it reads:

$$\frac{\hat{\mathcal{F}}^t_n - \mathcal{F}^t_n}{\Delta t} + \frac{\mathcal{F}^x_{n+1/2} - \mathcal{F}^x_{n-1/2}}{\Delta x} = 0, \tag{15}$$

where $\mathcal{F}^t$ and $\mathcal{F}^x$ are t- and x-component of the 'flux vector' in the laboratory frame, Δt, Δx are stepsizes, and $\hat{}$ denotes an implicit (unknown) time layer.

In terms of the primitive variables, the components of $\mathcal{F}^t$ have the following form:

$$\mathcal{F}^t = \left\{ \begin{array}{c} \gamma^2 w - p + \dfrac{\boldsymbol{E}^2 + \boldsymbol{H}^2}{8\pi} \\ w\gamma^2 \boldsymbol{v} + \boldsymbol{S} \\ 0 \\ \boldsymbol{H} \\ \rho\gamma \\ \rho s \gamma \end{array} \right\},$$

where $\boldsymbol{E} = -[\boldsymbol{v}, \boldsymbol{H}]$ is an electric field and $\boldsymbol{S} = [\boldsymbol{E}, \boldsymbol{H}]/4\pi$ is a Pointing flux in the laboratory frame. To calculate the fluxes in cells' interfaces $n + 1/2$ the formula (12) is used, where the left and right states are adopted as $\mathcal{U}_n$ and $\mathcal{U}_{n+1}$.

The system (15) is overposed since it contains ten equations in eight unknown variables $\boldsymbol{v}, \boldsymbol{H}, p, s$. We adopt the set 1–4, 6–9 (i.e. equations of energy, momentum, induction and continuity) as independent equations.[2] It is necessary to stress that earlier we used the entropy conservation equation (last equation of equations 5) for system closure, to get the Jacobian and to build an approximate Riemann solver. After the approximate Riemann solver has been built we don't need the entropy conservation equation and use the energy equation instead, which permits us to treat the shocks and the entropy production at the shocks correctly. As for maintaining constraints $u_i u^i = -1$ and $u_i h^i = 0$, this will be explicitly accounted for when calculating the 'primitive' variables (see Section 8).

8 CALCULATION OF PRIMITIVE VARIABLES

Using (15) we can calculate the values of

$$\rho\gamma = A, \quad w\gamma^2 \boldsymbol{v} + \boldsymbol{S} = \boldsymbol{B},$$

$$\boldsymbol{H}, \quad \gamma^2 w - p + \frac{\boldsymbol{E}^2 + \boldsymbol{H}^2}{8\pi} = C \tag{16}$$

on the implicit time layer. To calculate the primitive variables $\{u^i, h^i, p, s\}$ or, equivalently, $\{\boldsymbol{v}, \boldsymbol{H}, p, s\}$ we have to use some iterative process. Actually, it is enough to calculate $Z = \gamma^2 w$; then the remaining variables are expressed explicitly. System (16) is reduced to a single equation in Z:

$$f(Z) = Z - \frac{\Gamma - 1}{\Gamma}\left(\frac{Z}{\gamma^2} - \frac{A}{\gamma}\right) - \left[\frac{\boldsymbol{H}^2}{\gamma^2} + \frac{(\boldsymbol{B}\cdot\boldsymbol{H})^2}{Z^2}\right] = C - \frac{\boldsymbol{H}^2}{4\pi}, \tag{17}$$

where the Lorentz factor $\gamma(Z)$ is defined as

$$\frac{1}{\gamma^2} = 1 - \boldsymbol{v}^2 = 1 - \left[\boldsymbol{B} + \frac{(\boldsymbol{B}\cdot\boldsymbol{H})}{4\pi Z}\boldsymbol{H}\right]^2 \Big/ \left(Z + \frac{\boldsymbol{H}^2}{4\pi}\right)^2. \tag{18}$$

According to this formula $1/\gamma^2 \leq 1$. Additionally, we should require $1/\gamma^2 > 0$ but we impose a stronger condition $p \geq 0$ to obtain

$$Z^2/\gamma^2 \geq A^2. \tag{19}$$

[2] Basically, the choice of set (2–4, 6–10) (i.e. using the entropy equation instead of the energy equation) is also possible. This approach can be advisable for the simulation of flows without shocks.

This inequality is valid if $Z \geq Z_*$ where Z_* is a maximal real root of the fourth-order equation $Z^2/\gamma^2 = A^2$. Since $f(Z)$ in the left-hand side of equation (17) is a monotonically increasing function, the condition for solvability of equation (17) with the constraint (19) is $f(Z_*) \leq C - \boldsymbol{H}^2/4\pi$. The solution of equation (17) can be found, for instance, by the Newton method.

9 RESULTS OF TEST CALCULATIONS

To check how the scheme works we have calculated a few test problems. For all test runs a uniform spatial grid with 200 cells and with the stepsize $\Delta x = 1$ was used. The value of the time-step was adopted as $\max|\lambda|\tau/h = \kappa_0$ with Courant number $\kappa_0 = 0.9$.[3] The value of the adiabatic index was adopted as $\Gamma = 1.4$. Figs 2–5 depict the exact solution (a solid line)[4] as calculated by our scheme (symbols).

9.1 Test 1: fast shock wave

A fast shock wave propagates to the right with the velocity $V_{\mathrm{F}} = 0.1$. The background (right state) and post-shock (left state) parameters are

$$\rho_{\mathrm{L}} = 184.86, \quad \rho_{\mathrm{R}} = 91.018,$$
$$p_{\mathrm{L}} = 2.8404, \quad p_{\mathrm{R}} = 1.0046,$$
$$v_{x\mathrm{L}} = 0, \quad v_{x\mathrm{R}} = -0.102,$$
$$v_{y\mathrm{L}} = 0, \quad v_{y\mathrm{R}} = 0.0068,$$
$$v_{z\mathrm{L}} = v_{z\mathrm{R}} = 0,$$
$$H_{y\mathrm{L}} = 2.089, \quad H_{y\mathrm{R}} = 0.9839,$$
$$H_{z\mathrm{L}} = H_{z\mathrm{R}} = 0.$$

The value of longitudinal magnetic field was adopted as $H_x = H_{x0} = 1.5$. The initial position of the wave is at the interface between the 10th and 11th cells. Results of the test run are presented in Fig. 2.

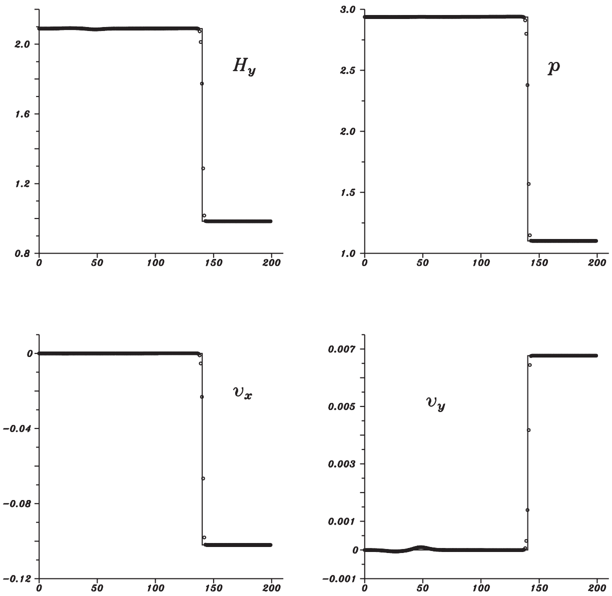

Figure 2. A fast MHD shock. Left top panel – y-component of the magnetic field H_y, right top panel – pressure p, left bottom panel – x-component of the velocity v_x, right bottom panel – y-component of the velocity v_y. The exact solution is drawn by a solid line, and the approximate solution by symbols.

9.2 Test 2: slow shock wave

A slow shock wave propagates to the right with the velocity $V_{\mathrm{S}} = 0.1$. The background (right state) and post-shock (left state) parameters are

$$\rho_{\mathrm{L}} = 3.039, \quad \rho_{\mathrm{R}} = 0.805,$$
$$p_{\mathrm{L}} = 0.4428, \quad p_{\mathrm{R}} = 0.0366,$$
$$v_{x\mathrm{L}} = 0, \quad v_{x\mathrm{R}} = -0.206,$$
$$v_{y\mathrm{L}} = 0, \quad v_{y\mathrm{R}} = -0.547,$$
$$H_{y\mathrm{L}} = 0.995, \quad H_{y\mathrm{R}} = 3.005.$$

The values of other parameters are the same as in Test 1. Results of the run are presented in Fig. 3.

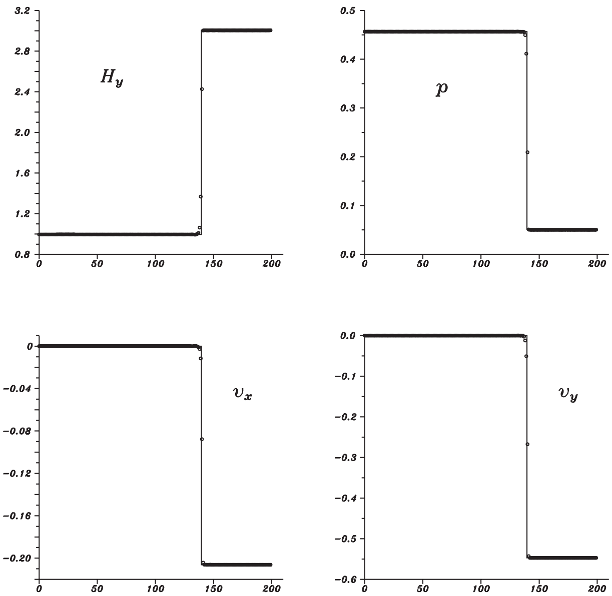

Figure 3. A slow MHD shock. Left top panel – y-component of the magnetic field H_y, right top panel – pressure p, left bottom panel – x-component of the velocity v_x, right bottom panel – y-component of the velocity v_y. The exact solution is drawn by a solid line, and the approximate solution by symbols.

9.3 Test 3: Alfvén wave

We have considered the propagation of an Alfvén wave (Komissarov 1997). In a simple Alfvén wave all variables $\mathcal{U}$ depend on the phase angle $\phi(x)$ only. The dispersional equation for $\pm A$ waves reads:

$$\sqrt{w + h^2}u^i \frac{\partial \phi}{\partial x^i} = \pm h^i \frac{\partial \phi}{\partial x^i},$$

in this wave p, s, h^2 being constant, and four-vectors of magnetic field and velocity being connected by a relation

$$h^i = \pm\sqrt{w + h^2}u^i + a^i,$$

[3] Excepting test 4.
[4] Excepting Fig. 5.

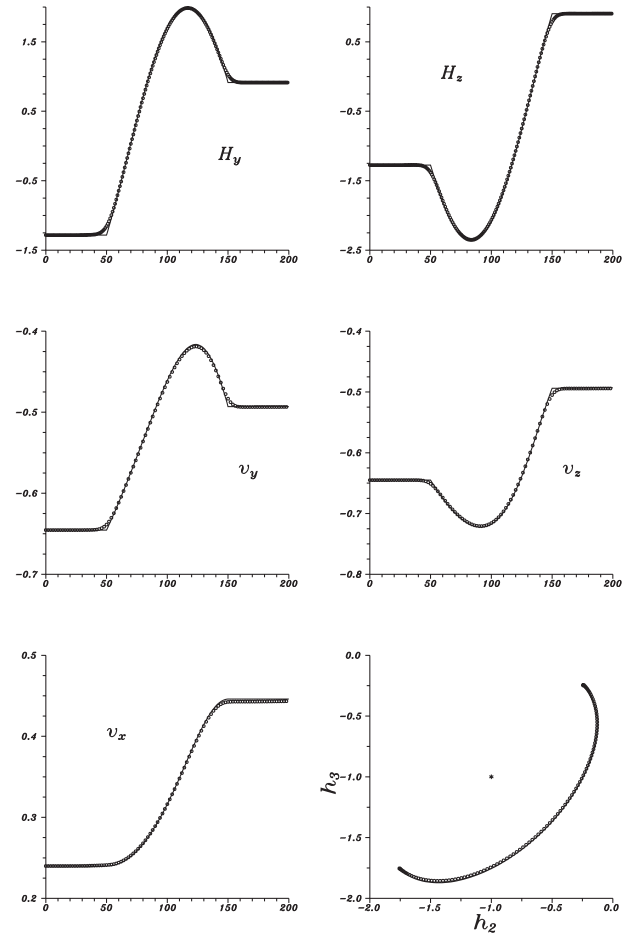

Figure 4. An Alfvén wave. Left top panel – y-component of the magnetic field H_y, right top panel – z-component of the magnetic field H_z, left middle panel – y-component of velocity v_y, right middle panel – z-component of velocity v_z, left bottom panel – x-component of velocity v_x, right bottom panel – a locus of the points with (h_2, h_3) coordinates in the Alfvén wave. The exact solution is drawn by a solid line, and the approximate solution by symbols.

where $a^i =$ constant is a time-like vector and $a^2 = a_i a^i = -w$. Besides this we have $a^i \partial\phi/\partial x^i = 0$.

If this wave is plane (i.e. in some reference frame the phase angle depends on $x^0 = t$ and $x^1 = x$ only), then the last equation gives for this reference frame $\phi = \phi(x - \lambda t)$ where $\lambda = -a^1/a^0 =$ constant.

The evolution of the four-vector of velocity in the simple Alfvén wave is described by

$$\frac{\mathrm{d}u^i}{\mathrm{d}\phi} = \pm A(\phi)\varepsilon^{ijkl} U_j \mu_k a_l, \tag{20}$$

where $\mu_k = (-\lambda, 1, 0, 0)$, and $A(\phi)$ is an arbitrary function of the phase angle. System (20) has three integrals:

$$\left.\begin{aligned} &u_i u^i = -1, \\ &a_i u^i = \pm\sqrt{w + h^2}, \\ &\mu_i u^i = \alpha = \text{constant}, \end{aligned}\right\} \tag{21}$$

and the dependence of the solution on the phase angle is defined by (an arbitrary) function $A(\phi)$. Equations (21) define an ellipse on the

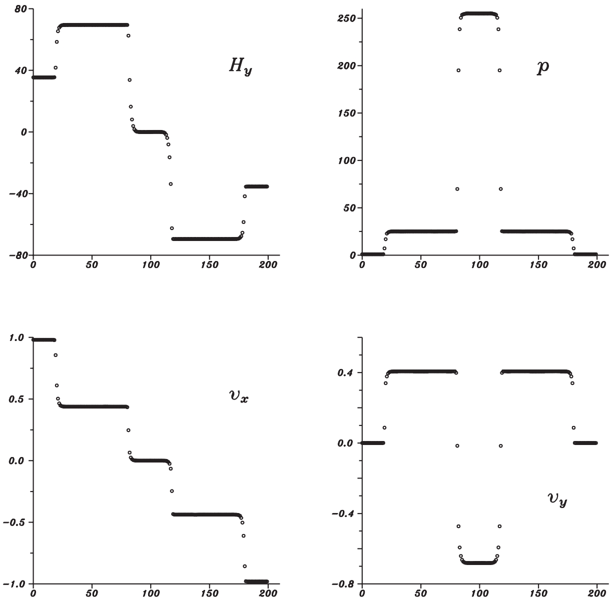

Figure 5. A shock-tube test from Komissarov (1999) (collision of flows with opposite orientation of the perpendicular components of the magnetic field). The Courant number was adopted as $\kappa_0 = 0.3$. Left top panel – y-component of magnetic field H_y, right top panel – pressure p, left bottom panel – x-component of velocity v_x, right bottom panel – y-component of velocity v_y. The approximate solution is drawn by symbols.

(u_y, u_z) plane, its shape being described as

$$(w + a_3^2)\eta^2 - 2a_2 a_3 \eta\zeta + (w + a_2)\zeta^2 = h^2(a_0^2 - a_1^2)/w - \alpha^2 a_0^2, \tag{22}$$

where

$$\eta = u_y - a_2\sqrt{w + h^2}/w,$$

$$\zeta = u_z - a_3\sqrt{w + h^2}/w.$$

To build a solution for the plane Alfvén wave we adopt the following parameters: enthalpy $w = 1$; pressure $p = 0.2$; components of four-vector a^i are

$$a^0 = \sqrt{w + \boldsymbol{a}^2} = 2, \quad \boldsymbol{a} = (1, 1, 1),$$
$$a^2 = a_i a^i = -w;$$

magnetic field $h^2 = 1$; the value of the parameter α is

$$\alpha = 0.9\sqrt{h^2/(1 - a_1^2/a_0^2)w}.$$

Taking these values we can determine $u_y(x, t = 0)$ and $u_z(x, t = 0)$ that satisfy equation (22) and all the other variables as well.

Fig. 4 presents profiles H_y, H_z, v_y, v_z, v_x for $t = 135.6$. The right bottom panel of Fig. 4 presents a locus of the points with (h_2, h_3) coordinates in the Alfvén wave for the same moment of time (as usual, the exact solution is drawn by a solid line, and the approximate solution by symbols).

9.4 Test 4: test from Komissarov (1999)

We have also run a test described by Komissarov (1999) – a collision of flows with opposite orientation of the perpendicular components of the magnetic field. For this run the Courant number was adopted as $\kappa_0 = 0.3$. We consider a shock-tube problem with the left and right states as

$$\rho_L = \rho_R = 1,$$
$$p_L = p_R = 1,$$
$$v_{xL} = \frac{V}{\sqrt{1+V^2}}, \quad v_{xR} = -\frac{V}{\sqrt{1+V^2}}, \quad V = 0.5,$$
$$v_{yL} = V, \quad v_{yR} = -V,$$
$$v_{zL} = v_{zR} = 0,$$
$$H_{xL} = H_{xR} = H_{x0} = 10\sqrt{4\pi},$$
$$H_{yL} = H_{yR} = 10\sqrt{4\pi},$$
$$H_{zL} = H_{zR} = 0.$$

The picture of the flow arose from decay of the discontinuity and is symmetrical with respect to its initial position $x = 0$ and consists of the left- and right-propagating shocks. Results of the test run are presented in Fig. 5.

10 CONCLUSIONS

We have constructed a Godunov-type scheme for ideal relativistic MHD. The scheme is based on the approximate solution of the Riemann problem. We have obtained explicit expressions for the wave amplitude resulting from the decay of discontinuity in a form suitable for analysis in degenerate cases when: (i) the component of the front perpendicular to the magnetic field vanishes; (ii) the component of the front parallel to the magnetic field vanishes. It has been shown that this approach permits us to eliminate the degeneracies, i.e. to obtain fluxes of conservative variables independently of the way in which we calculate values that are undefined in degenerate cases.

We have run a set of test calculations and found the scheme is efficient and accurate for simulation of flows with rather intensive fast and slow MHD shocks as well as with alfvénic waves. In the future we plan to extend our scheme for 2D case and to apply it to simulation of relativistic MHD flows in astrophysical objects.

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REFERENCES

Anile A. M., 1989, Relativistic Fluids and Magnetofluids. Cambridge Univ. Press, Cambridge
Anile A. M., Pennisi S., 1987, Ann. Inst. Henri Poincaré, 46, 27
Appenzeller I., Mundt R., 1989, A&AR, 1, 291
Balsara D., 2001, ApJS, 132, 83
Begelman M. C., Blandford R. D., Rees M. J., 1984, Rev. Mod. Phys., 56, 255
Bhacall J. N., Kihakos S., Schneider D. P., Davis R. J., Muxlow T. W. B., Garrington S. T., Conway R. G., Unwin S. C., 1995, ApJ, 452, L91
Biretta J. A., Zhou F., Owen F. N., 1995, ApJ, 127, 582
Blandford R. D., Netzer H., Woltjer L., 1991, Active Galactic Nuclei. Springer-Verlag, Berlin
Bogovalov S. V., 1997, A&A, 327, 662
Boissé P., Le Brun V., Bergeron J., Deharveng J. M., 1998, A&A, 333, 841
Brio M., Wu C. C., 1988, J. Comput. Phys., 75, 400
Camenzind M., 1990, in Klare G., ed., Magnetized Disk–Winds and the Origin of Bipolar Outflows. Kluwer Academic, Dordrecht, p. 234
Dey A., van Breugel W. J. M., 1994, AJ, 107, 1977
Dubal M. R., Pantano O., 1993, MNRAS, 261, 203
Einfeldt B., 1988, SIAM J. Numer. Anal., 25, 294
Fendt C., Camenzind M., Appl S., 1995, A&A, 300, 791
Godunov S. K., 1959, Math. Sbornik, 89, 47, 271
Goméz J.-L., 2001, preprint (astro-ph/0109338)
Khanna R., Camenzind M., 1996, A&A, 307, 665
Koldoba A. V., Ustyugova G. V., 1999, preprint KIAM N 67, Moscow (in Russian)
Komissarov S. S., 1997, Phys. Lett. A, 232, 435
Komissarov S. S., 1999, MNRAS, 303, 343
Landau L. D., Lifshitz E. M., 1959, Fluid Mechanics. Pergamon Press, London
Levinson A., Blandford R. D., 1996, ApJ, 456, L29
Lichnerowicz A., 1967, Relativistic Hydrodynamics and Magnetohydrodynamics. Benjamin Press, New York
Martí J., Müller E., 1999, Living Rev. in Relativity, 2, No. 3 (http://www.livingreviews.org/Articles/Volume2/1999-3marti)
Mészáros P., Rees M. J., 1992, MNRAS, 258, 41
Mirabel I. F., Rodrígeez L. F., 1994, Nat, 371, 46
Mirabel I. F., Rodrígeez L. F., 1996, in Tsiganos K., ed., Solar and Astrophysical Magnetohydrodynamic Flows. Kluwer Academic, Dordrecht, p. 693
Montmerle T., Feigelson E. D., Bouvier J., Andreé P., 1993, in Levy E. H., Lunine J. I., eds, Magnetic Fields, Activity and Circumstellar Material around Young Stellar Objects. Kluwer Academic, Dordrecht, p. 689
Osher S., Solomon F., 1982, Math. Comp., 38, 339
Rees M. J., Mészáros P., 1994, ApJ, 430, L93
Roe P. L., 1986, Ann. Rev. Fluid Mech., 18, 337
Ryu D., Jones T. W., 1995, ApJ, 452, 228
van Putten M. H. P. M., 1991, Commun. Math. Phys., 141, 63
van Putten M. H. P. M., 1993, J. Comput. Phys., 105, 339
van Putten M. H. P. M., 1994, in Brown J. D., Chu M. T., Ellison D. C., Plemmons R. J., eds, Proc. Cornelius Lanczos Int. Centenary Conf.. SIAM, Philadelphia, p. 449
van Putten M. H. P. M., 1995, SIAM J. Numer. Anal., 32, 1504
van Putten M. H. P. M., 1996, ApJ, 467, L57
van Putten M. H. P. M., 1998, preprint (astro-ph/9804139)

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