Summer Project

*Fields Generated by an Oscillating Dipole*

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Chapter 1

The Free (Isolated) Dipole

1.1 Introduction & Statement of the Problem

In this report we will discuss the problem of calculating the fields produced by an arbitrarily oriented oscillating dipole. We will first treat the isolated dipole, and then consider the case when a surface is added to the picture. The former case will be investigated using three major expansions: spherical, cylindrical and plane wave expansions. Being somewhat more involved, the latter case will be treated only with cylindrical and plane wave expansions. You will be able to find most of the derivations in this report...

1.2 Mathematical Notes

Before going into the derivations and calculations it could be beneficial to review some basic mathematical aspects related to the problem at hand. In this section we shall do just this...

1.2.1 Spherical, Cylindrical and Plane Waves

Alright! We shall be doing three kinds of wave expansions which were already listed in the title. But what do these mean? Without really understanding the properties of the waves into which we will expand our fields, it is highly unlikely that we get through...

The three types of waves are classified according to their wavefronts, so to speak. Here a wavefront is defined such that at all points on the wavefront the phase of the field (the electric field, for instance) is constant. That is to say, for a spherical wave the phase is constant on a sphere centered at some point (which is probably the source), and so on. However, it is only the phase that is constant! We do not make any statements about the amplitude at various points on the surface defining the wavefront. In fact, it will be the case that the amplitude will be varying in space (even on a wavefront) for the cylindrical and plane waves. That is why we should denote the amplitudes of the three kinds of waves with $S_0(r)$, $C_0(r)$, and $P_0(r)$, respectively.

In all mathematics that will soon follow we will make use of the complex representation of the electric fields. It is left to you to verify that the following are the general forms of the spherical, cylindrical and plane waves, respectively:

$$S(r, t) = S(r) \exp(-i\omega t) = S_0(r) \exp(ikr) \exp(-i\omega t)$$
$$C(r, t) = C(r) \exp(-i\omega t) = C_0(r) \exp(ik || \rho) \exp(-i\omega t)$$
$$P(r, t) = P(r) \exp(-i\omega t) = P_0(r) \exp(i \mathbf{k} \cdot \mathbf{r}) \exp(-i\omega t)$$
where $\rho = \sqrt{x^2 + y^2}$ and $k_{||} = \sqrt{k_x^2 + k_y^2}$. The definitions of $r$, $k$, etc should be obvious.

Defined in this way, the spherical waves are centered at the origin; the cylindrical waves are centered around the $z$ axis; and (forgive me but again obviously) the plane waves are centered nowhere.

That is how a good homo sapiens sapiens would define the three types of wavefronts. Unfortunately, there are some surviving homo sapiens neanderthalis among us (including me myself) who would name a totally irrelated type of waves rather unfairly as cylindrical. So, forgetting about the above definition, we state without proof that “cylindrical waves are given in the following general form:"

$$C(r, t) = C(r) \exp(-i\omega t) = C_0(r)J_0(k_{||}\rho) \exp(ik_z|z|) \exp(-i\omega t)$$

It takes no more than two months to realize that these are nothing but plane waves; not general at all, but rather very particular plane waves parallel to the $x - y$ plane.

But physicists do nothing without a justification. A reason why such waves could be termed as cylindrical could be that the amplitude part consisting of $J_0(k_{||}\rho)$ is constant on a cylindrical surface. Still, in my own opinion, this is a very wrong name, misleading, and concealing very crucial facts...

### 1.2.2 Dyads

Before beginning our discussion it is important to get used to the notation that will be used throughout the text. Firstly, we will denote a vector in the following form:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

and its transpose:

$$v^T = (v_1 \ v_2 \ v_3)$$

The usual scalar dot product of two vectors will be denoted by:

$$v \cdot u = v^T u = v_1u_1 + v_2u_2 + v_3u_3$$

Notice that there is one more possible type of product between two vectors, and that is:

$$vu^T = \begin{pmatrix} v_1u_1 & v_1u_2 & v_1u_3 \\ v_2u_1 & v_2u_2 & v_2u_3 \\ v_3u_1 & v_3u_2 & v_3u_3 \end{pmatrix}$$

The resulting $3 \times 3$ matrix is the so-called dyad. The following equality involving the two types of products just described holds for any three vectors:

$$a^T (bc^T) = (a^T b) c^T$$

You may verify that the equation is indeed true yourself.
1.3 Theory: The Dipole Electromagnetic Field

We shall start with reviewing the Maxwell equations. Recall that there are two distinct sets of equations: the microscopic and the macroscopic Maxwell equations. These two sets are, respectively:

\[
\begin{align*}
\varepsilon_0 \nabla \cdot E &= \rho \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \cdot B &= 0 \\
\frac{1}{\mu_0} \nabla \times B &= J + \varepsilon_0 \frac{\partial E}{\partial t}
\end{align*}
\]  
\(1'1\)

\[
\begin{align*}
\nabla \cdot D &= \rho \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \cdot B &= 0 \\
\nabla \times H &= J + \frac{\partial D}{\partial t}
\end{align*}
\]  
\(1'2\)

We have written the equations in the SI units. Throughout the text we shall stick to this system of units...

The sources obey the continuity equation:

\[
\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0
\]  
\(1'3\)

This equation follows directly from the law of conservation of charge. Note that in the equation there are four unknowns in total: the charge density \(\rho\) and the three components of the current density \((J_x, J_y, J_z)\). We can remove one of these by defining a polarization field in the following way:

\[
\rho = -\nabla \cdot P \quad \text{and} \quad J = \frac{\partial P}{\partial t}
\]  
\(1'4\)

Obviously, \(P\) satisfies the continuity equation automatically.

Often it is convenient to express the electric and magnetic fields in terms of the vector and scalar potentials:

\[
\begin{align*}
E &= -\nabla \phi - \frac{\partial A}{\partial t} \\
B &= \nabla \times A
\end{align*}
\]  
\(1'5\)

There is some arbitrariness (or freedom) in the choice of these potentials due to the one extra component: the two potentials introduce four elements whereas only three would be sufficient. In other words, an additional arbitrary constriction can be imposed on the potentials. The constriction itself is referred to as a gauge. We shall make use of the so-called Lorentz gauge, which can be expressed in the following form:

\[
\nabla \cdot A + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0
\]  
\(1'6\)

This equation allows for the elimination of the scalar potential:

\[
\begin{align*}
\phi(r, t) &= -c^2 \int_0^t dt' \nabla \cdot A(r, t') \\
&= -c^2 \nabla^T \int_0^t dt' A(r, t')
\end{align*}
\]  
\(1'7\)
Now we can solve for the vector potential by going back to the Maxwell equations:

\[
\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\]

\[
\frac{1}{\mu_0} \nabla \times (\nabla \times \mathbf{A}) = \frac{\partial \mathbf{P}}{\partial t} - \varepsilon_0 \frac{\partial}{\partial t} \left( \nabla \phi + \frac{\partial \mathbf{A}}{\partial t} \right)
\]

\[
= \frac{\partial \mathbf{P}}{\partial t} + \varepsilon_0 \frac{\partial}{\partial t} \left( c^2 \nabla \nabla^T \int_0^t dt' \mathbf{A}(\mathbf{r}, t') - \frac{\partial \mathbf{A}}{\partial t} \right)
\]

At this point we give a short break to remind the reader of an important vector equation (true in Cartesian coordinates only):

\[
\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla) \mathbf{A}
\]

\[
= (\nabla \nabla^T - \nabla^T \nabla) \mathbf{A}
\]

(18)

The derivation of this important relation is rather easy if one makes use of the Kronecker delta and the Levi-Civita tensor notations, together with the so-called Einstein (summation) convention:

\[
[\nabla \times (\nabla \times \mathbf{A})]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times \mathbf{A})_k = \varepsilon_{ijk} \varepsilon_{klm} \frac{\partial}{\partial x_l} A_m = \frac{\partial^2 A_m}{\partial x_j \partial x_l} \varepsilon_{ijk} \varepsilon_{klm}
\]

\[
= \frac{\partial^2 A_m}{\partial x_j \partial x_l} \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} = \frac{\partial^2 A_j}{\partial x_j \partial x_l} \delta_{il} - \frac{\partial^2 A_m}{\partial x_j \partial x_l} \delta_{im} = \frac{\partial^2 A_j}{\partial x_j \partial x_l} \delta_{il} - \frac{\partial^2 A_m}{\partial x_j \partial x_l} \delta_{im}
\]

\[
= \frac{\partial^2 A_j}{\partial x_j \partial x_l} - \frac{\partial^2 A_i}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} A_j \right) - \left( \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} A_i \right) = \frac{\partial}{\partial x_i} (\nabla \nabla^T \mathbf{A}) - (\nabla^T \nabla) \mathbf{A}_i
\]

Q.E.D.

Having completed this short exercise we can now return to the derivation:

\[
\frac{1}{\mu_0} (\nabla \nabla^T \mathbf{A} - \nabla^T \nabla \mathbf{A}) = \frac{\partial \mathbf{P}}{\partial t} + \varepsilon_0 \left( c^2 \nabla \nabla^T \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right)
\]

After some rearrangement and through the application of the relation \( c^2 = 1/\mu_0 \varepsilon_0 \) we arrive at:

\[
\left( \nabla^T \nabla - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -\mu_0 \frac{\partial \mathbf{P}}{\partial t}
\]

(19)

the so-called dispersion relation for the vector potential \( \mathbf{A}(\mathbf{r}, t) \). The solution of this differential equation involves Green’s functions; but, we will omit this part and pass on to the solution directly:

\[
\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial \mathbf{P}(\mathbf{r}', t')}{\partial t} + \frac{\mu_0}{4\pi} \int d\mathbf{r}' e^{ik|\mathbf{r} - \mathbf{r}'|} \frac{\partial \mathbf{P}(\mathbf{r}', t)}{\partial t}
\]

(10)
where use has been made of the definition of the so-called *retarded time*:\(^1\)

\[ t' = t - \frac{|r - r'|}{c} \]  

(1'11)

If we assume a *time harmonic* polarization field, i.e.:

\[ \mathbf{P}(\mathbf{r}, t) = \mathbf{P}(\mathbf{r}) \exp(-i\omega t) \]  

(1'12)

where \( \omega = ck \), we simply obtain:

\[
\mathbf{A}(\mathbf{r}) = -\frac{i\omega \mu_0}{4\pi} \int d\mathbf{r}' \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \mathbf{P}(\mathbf{r}')
\]

\[
\mathbf{A}(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}) \exp(-i\omega t) \]  

(1'13)

From here we can proceed on to the calculation of the electric field. But before that we turn to the question of what the polarization field actually means. Recall that the dipole moment (or dipole strength) has dimensions of charge times length. It should be obvious from eq.(1'4) that the polarization field has dimensions of charge per area, or dipole moment per volume. The difference between the polarization field and the dipole moment (which we choose to denote by \( \mathbf{p} \)) is that the first is related to the continuum, and the second, to the discrete model. At this point perhaps it is a good idea to quickly review several important equations regarding these two physical concepts.

Suppose we divide the space into cells of volume \( \Omega_i \). The dipole moment of cell \( i \) is then given through:

\[
\mathbf{p}_i = \int_{\Omega_i} d\mathbf{r}' \mathbf{P}(\mathbf{r}')
\]  

(1'14)

This is a way to discretize a given system. In general, we can approximate the polarization field by introducing \( n \) dipoles. This can be done through the following substitution:

\[
\mathbf{P}(\mathbf{r}) = \sum_i \mathbf{p}_i \delta(\mathbf{r} - \mathbf{r}_i)
\]  

(1'15)

On the other hand, the electric field and the polarization field are related through the following equation:

\[
\mathbf{P}(\mathbf{r}) = \varepsilon_0 \chi(\mathbf{r}) \mathbf{E}(\mathbf{r})
\]

\[
\chi(\mathbf{r}) = \kappa(\mathbf{r}) - 1 = \varepsilon(\mathbf{r})/\varepsilon_0 - 1
\]  

(1'16)

where \( \chi(\mathbf{r}) \) is the *(electric) susceptibility*, \( \kappa(\mathbf{r}) \) is the *relative permittivity*, and \( \varepsilon(\mathbf{r}) \) is the *(electric) permittivity* of the medium. Apparently:

\[
\mathbf{D}(\mathbf{r}) = \varepsilon(\mathbf{r}) \mathbf{E}(\mathbf{r}) = \varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})
\]  

(1'17)

Now we are ready to tackle the problem of calculating the electric field from the vector potential as given in eq.(1'13). We make use of eqs.(1'5) and (1'7):

\[
\mathbf{E}(\mathbf{r}, t) = c^2 \nabla \nabla^T \int dt' \mathbf{A}(\mathbf{r}, t') - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}
\]

\[
= c^2 \nabla \nabla^T \int dt' \left( \frac{\mu_0}{4\pi} \int d\mathbf{r}' \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial \mathbf{P}(\mathbf{r}', t')}{\partial t'} \right) - \frac{\partial}{\partial t} \left( \frac{\mu_0}{4\pi} \int d\mathbf{r}' \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial \mathbf{P}(\mathbf{r}', t)}{\partial t} \right)
\]

\[
= \frac{1}{4\pi\varepsilon_0} \int d\mathbf{r}' \left( \nabla \nabla^T - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \mathbf{P}(\mathbf{r}', t)
\]

\(^1\)Take care not to mix up the retarded time with some other time definitions that will follow. Below we will denote by the same symbol (i.e. by \( t' \)) another temporal variable.
or, simply:

\[
E(r) = \frac{1}{4\pi\epsilon_0} \int d\mathbf{r'} \left( \nabla \nabla^T + k^2 \right) \frac{e^{ik|r-r'|}}{|r-r'|} \mathbf{P}(\mathbf{r'})
\]  

(1’18)

Note that the differential operators act only on \( r \) and not on \( r' \). Thus we have obtained the continuum formulation of the electric field. We can rewrite the above equation in a new symbolic form:

\[
E(r) = \int d\mathbf{r'} t(\mathbf{r}, \mathbf{r'}) \mathbf{P}(\mathbf{r'})
\]

where \( t(\mathbf{r}, \mathbf{r'}) \) is called the transfer kernel. To switch on to the discrete formulation involving, say, \( n \) dipoles we make use of eq.(1’15) to obtain:

\[
E(r) = \sum_i T(r, r_i) \mathbf{p}_i
\]

\[
T(r, r_i) = \frac{1}{4\pi\epsilon_0} \left( \nabla \nabla^T + k^2 \right) \frac{e^{ik|r-r_i|}}{|r-r_i|}
\]

(1’19)

where \( T(r, r_i) \) is the so-called transfer tensor. In the oscillating dipole case we will have \( n = 1 \), i.e. will deal with a discrete system, and shall examine the transfer tensor in some detail...

1.4 The Second Theory: Non-vacuum Media

We are not done yet. We have only derived the equations for a vacuum. Now we shall do the same for a general medium. We begin by rewriting the Maxwell equations, eq.(1’1):

\[
\varepsilon \nabla \cdot \mathbf{E} = \rho \quad \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad \quad \frac{1}{\mu} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}
\]

(1’20)

where \( \varepsilon \) and \( \mu \) are the permittivity and the permeability of the medium. The wave number in the medium becomes accordingly: \( k = \omega \sqrt{\varepsilon\mu} \).

The continuity equation, the polarization field definition, and the relations between the fields and potentials given by eqs.(1’3), (1’4) and (1’5) still hold true. But the Lorentz gauge should have a new definition:

\[
\nabla \cdot \mathbf{A} + \frac{1}{v^2} \frac{\partial \phi}{\partial t} = 0
\]

(1’21)

where \( v \) is the speed of light in the new medium and is given through \( v^2 = 1/\mu\varepsilon \). So, all we have to do is actually to replace \( c, \epsilon_0 \) and \( \mu_0 \) by \( v, \varepsilon \) and \( \mu \) in our derivations. The vector potential simply becomes (we consider only time-harmonic fields):

\[
\mathbf{A}(\mathbf{r}, t) = \frac{\mu}{4\pi} \int d\mathbf{r'} e^{ik|r-r'|} \frac{\partial \mathbf{P}(\mathbf{r'}, t)}{|\mathbf{r}-\mathbf{r'}|} \frac{\partial \mathbf{P}(\mathbf{r'}, t)}{\partial t}
\]

\[
= -\frac{i\omega\mu}{4\pi} \int d\mathbf{r'} e^{ik|r-r'|} \mathbf{P}(\mathbf{r'}) \exp(-i\omega t)
\]

(1’22)
The electric field can be calculated in much the same way as before:

\[
E(r, t) = v^2 \nabla \nabla^T \int dt' A(r, t') \frac{\partial A(r, t)}{\partial t} \\
= \frac{1}{4\pi \varepsilon} \int dr' \left( \nabla \nabla^T + k^2 \right) \frac{e^{ik|r-r'|}}{|r-r'|} P(r') \exp(-i\omega t) \\
= \frac{i}{\omega \varepsilon \mu} \left( \nabla \nabla^T + k^2 \right) A(r, t)
\] (1'23)

It is possible to define other potentials as well. One of these is the Hertz potential, which is denoted by \( \Pi(r, t) \) and is used in the literature (by Novotny\(^1\), in particular). The following equations hold:

\[
E = (\nabla \nabla^T + k^2) \Pi \\
H = -i\omega \varepsilon \nabla \times \Pi
\] (1'24)

For time-harmonic fields the relation between the Hertz potential and the vector potential is found to be\(^2\):

\[
\nabla \times \Pi = \frac{i}{\omega \varepsilon} H = \frac{i}{\omega \varepsilon \mu} B = \frac{i}{\omega \varepsilon \mu} \nabla \times A \\
\Pi = \frac{i}{\omega \varepsilon \mu} A
\] (1'25)

Note that the only difference between the Hertz potential and the vector potential is actually in a multiplicative constant. The differential equation determining the Hertz potential can therefore be written directly from eq.(1'9), the differential equation for the vector potential\(^3\):

\[
\left( \nabla^T \nabla + k^2 \right) \Pi = -\frac{i}{\omega \varepsilon} \frac{\partial P}{\partial t}
\] (1'26)

The solution of this equation is analogous to that for the vector potential, so we have:

\[
\Pi(r, t) = \frac{1}{4\pi \varepsilon} \int dr' e^{ik|r-r'|} \frac{P(r') \exp(-i\omega t)}{|r-r'|}
\] (1'27)

For an oscillating dipole this reduces to:

\[
\Pi(r) = \frac{1}{4\pi \varepsilon} \frac{\exp(ik|r-r'|)}{|r-r'|} P_j
\] (1'28)

The Hertz potential will be important for us when we compare our results with Novotny’s...

\(^2\)Novotny has probably made a mistake by claiming that the relation in question is:

\[
\Pi = i\omega \frac{\varepsilon \mu}{\varepsilon_0 \mu_0} A
\]

(Note that he denotes by \( \varepsilon \) and \( \mu \) the relative permittivity and permeability. That is why, in our notation, we had to make the division by the free space parameters.)

\(^3\)By claiming that the Hertz potential satisfies the vector Helmholtz equation:

\[
\left( \nabla^T \nabla + k^2 \right) \Pi = 0
\]

Novotny actually solves the source free field. It is not quite understandable to me how Novotny could use the solutions to this free space equation given in the following form (in cylindrical coordinates):

\[
J_n(k_0 \rho) \exp(ikz + in\phi)
\]

for the “dipole on top of a surface” case. In fact, in his paper he quotes the correct solution, but later ignores it altogether.
1.5 Spherical Wave Expansion

Suppose that we have a free oscillating dipole of strength $\mathbf{p}_j$, located at position $\mathbf{r}_j$ in space. We are interested in the resultant electric field at point $\mathbf{r}_i$. (Sometimes we will denote the position vectors by $\mathbf{r}'$ and $\mathbf{r}$, respectively.) The electric field and the transfer tensor are given through eq.(1’19):

$$E(\mathbf{r}_i) = T(\mathbf{r}_i, \mathbf{r}_j) \mathbf{p}_j$$

$$T(\mathbf{r}_i, \mathbf{r}_j) = \frac{1}{4\pi\varepsilon_0} \left( \nabla\nabla^T + k^2 \right) \frac{e^{ik|\mathbf{r}_i-\mathbf{r}_j|}}{|\mathbf{r}_i - \mathbf{r}_j|}$$

After carrying out the differentiations in the transfer tensor explicitly we obtain the following result:

$$T(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{ikr_{ij}}}{4\pi\varepsilon_0} \left[ \left( 3\frac{r_{ij}^3}{r_{ij}^2} - 3ik\frac{r_{ij}^2}{r_{ij}} - k^2 \right) \hat{\mathbf{r}}_ij\hat{\mathbf{r}}^T + \left( -\frac{1}{r_{ij}^3} + \frac{ik}{r_{ij}^2} + \frac{k^2}{r_{ij}} \right) \mathbf{1} \right] \quad (1’29)$$

where $r_{ij} = \mathbf{r}_i - \mathbf{r}_j$. The derivation of this important result will be given below...

1.5.1 General Derivation of the Spherical Wave Expansion

First notice that we have:

$$\nabla f(\mathbf{r}) = \frac{\partial}{\partial r} f(\mathbf{r}) \hat{\mathbf{r}}$$

$$\nabla \mathbf{r}^T = \hat{\mathbf{x}}\hat{\mathbf{x}}^T + \hat{\mathbf{y}}\hat{\mathbf{y}}^T + \hat{\mathbf{z}}\hat{\mathbf{z}}^T = \mathbf{1}$$

We begin with applying the gradient (we use $r$ instead of $r_{ij}$):

$$\nabla \left( \frac{e^{ikr}}{r} \right) = e^{ikr} \left( \frac{ik}{r} - \frac{1}{r^2} \right) \hat{\mathbf{r}}$$

$$\nabla \nabla^T \left( \frac{e^{ikr}}{r} \right) = \nabla \left[ \nabla \left( \frac{e^{ikr}}{r} \right) \right]^T$$

$$= \nabla \left[ e^{ikr} \left( \frac{ik}{r^2} - \frac{1}{r^3} \right) \mathbf{r}^T \right]$$

$$= \nabla \left[ e^{ikr} \left( \frac{ik}{r^2} - \frac{1}{r^3} \right) \right] \mathbf{r}^T + e^{ikr} \left( \frac{ik}{r^2} - \frac{1}{r^3} \right) \nabla \mathbf{r}^T$$

$$= \frac{\partial}{\partial r} \left[ e^{ikr} \left( \frac{ik}{r^2} - \frac{1}{r^3} \right) \right] \hat{\mathbf{r}}^T + e^{ikr} \left( \frac{ik}{r^2} - \frac{1}{r^3} \right) \mathbf{1}$$

$$= e^{ikr} \left( \frac{3}{r^3} - \frac{3ik}{r^2} - \frac{k^2}{r} \right) \hat{\mathbf{r}}^T + e^{ikr} \left( \frac{ik}{r^2} - \frac{1}{r^3} \right) \mathbf{1}$$

We can now calculate the transfer tensor:

$$T(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{ikr_{ij}}}{4\pi\varepsilon_0} \left[ \left( \frac{3}{r_{ij}^3} - \frac{3ik}{r_{ij}^2} - \frac{k^2}{r_{ij}} \right) \hat{\mathbf{r}}_ij\hat{\mathbf{r}}^T + \left( -\frac{1}{r_{ij}^3} + \frac{ik}{r_{ij}^2} + \frac{k^2}{r_{ij}} \right) \mathbf{1} \right]$$

According to this definition, the electric field is given by:

$$E(\mathbf{r}_i) = T(\mathbf{r}_i, \mathbf{r}_j) \mathbf{p}_j$$
1.6 Cylindrical Wave Expansion

We can use the Sommerfeld identity to expand a spherical wave into cylindrical waves:

\[
\frac{e^{ikr}}{r} = i \int_0^\infty dk_{||} J_0(k_{||}\rho)e^{ik_z|z-h|}
\]

where \( J_0 \) is the Bessel function of first kind of order zero, and:

\[
k_z = \sqrt{k^2 - k_{||}^2} \\
r = \sqrt{\rho^2 + (z - h)^2}
\]

Here we replace \( r_{ij} \) by \( r \).

If we suppose that the dipole is on the axis of cylindrical symmetry, (but not necessarily along the z axis) we have the following result for the electric field:

\[
\mathbf{E}(\mathbf{r}) = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk_{||} \frac{k_{||}}{k_z} \left[ \nabla \nabla^T + k^2 \right] J_0(k_{||}\rho)e^{ik_z|z-h|} \mathbf{p}_j
\]

1.6.1 General Derivation

We proceed by calculating the derivatives. First notice that for two functions \( f(\rho) \) and \( g(z) \) we have:

\[
\nabla f(\rho)g(z) = g(z)\nabla f(\rho) + f(\rho)\nabla g(z) \\
\nabla f(\rho) = \frac{df}{d\rho} \hat{\rho} \\
\nabla g(z) = \frac{dg}{dz} \hat{z}
\]

Since \( \rho = \sqrt{x^2 + y^2} \), we find the components as:

\[
\nabla \rho = \begin{pmatrix} \frac{d\rho}{dx} \\ \frac{d\rho}{dy} \\ 0 \end{pmatrix} = \begin{pmatrix} x/\sqrt{x^2 + y^2} \\ y/\sqrt{x^2 + y^2} \\ 0 \end{pmatrix} = \hat{\rho} \\
\n\nabla \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

Thus we obtain our final result:

\[
\nabla f(\rho)g(z) = g(z)\frac{df}{d\rho} \hat{\rho} + f(\rho)\frac{dg}{dz} \hat{z}
\]

We apply this to the gradient in the electric field equation:

\[
\nabla J_0(k_{||}\rho)e^{ik_z|z-h|} = e^{ik_z|z-h|} \frac{d}{d\rho} J_0(k_{||}\rho) \hat{\rho} \pm ik_z J_0(k_{||}\rho) e^{ik_z|z-h|} \hat{z}
\]

\[
= \left[ -k_{||} J_1(k_{||}\rho) \hat{\rho} \pm ik_z J_0(k_{||}\rho) \hat{z} \right] e^{ik_z|z-h|}
\]
where the plus sign holds for \( z > h \) and the minus sign for \( z < h \). In this derivation we have used the following relation:

\[
\frac{d}{dx} J_n(x) = -J_{n+1}(x) + \frac{n}{x} J_n(x)
\]

true for any Bessel function, see Abramowitz & Stegun, eq.(9.1.27). This equation becomes:

\[
\frac{d}{d\rho} J_0(k||\rho) = -k|| J_1(k||\rho) \\
\frac{d}{d\rho} J_1(k||\rho) = -k|| J_2(k||\rho) + \frac{1}{\rho} J_1(k||\rho)
\]

for the special values of \( n = 0 \) and \( n = 1 \). The second step is to take the transpose and differentiate once more:

\[
\nabla \nabla^T J_0(k||\rho) e^{ik_z|z-h|} = \nabla \left[ -k|| J_1(k||\rho) \hat{\rho}^T \pm ik_z J_0(k||\rho) \hat{z}^T \right] e^{ik_z|z-h|} \\
\nabla \left[ -k|| J_1(k||\rho) e^{ik_z|z-h|} \hat{\rho}^T \pm ik_z J_0(k||\rho) e^{ik_z|z-h|} \hat{z}^T \right] \\
= -k|| \left[ \nabla J_1(k||\rho) e^{ik_z|z-h|} \right] \hat{\rho}^T - k|| \left[ J_1(k||\rho) e^{ik_z|z-h|} \right] \nabla \hat{\rho}^T \\
\pm ik_z \left[ \nabla J_0(k||\rho) e^{ik_z|z-h|} \right] \hat{z}^T
\]

Notice that the combination \( \nabla \hat{z}^T \) vanishes since \( \hat{z}^T \) is a constant unit vector. Now we evaluate the expressions involving gradients. Beginning from the left we evaluate the first one:

\[
\nabla J_1(k||\rho) e^{ik_z|z-h|} = \left[ \left( -k|| J_2(k||\rho) + \frac{1}{\rho} J_1(k||\rho) \right) \hat{\rho} \pm ik_z J_1(k||\rho) \hat{z} \right] e^{ik_z|z-h|}
\]

And then the second:

\[
\nabla \hat{\rho}^T = \nabla (\rho^T / \rho) \\
= (\nabla 1 / \rho) \rho^T + 1 / \rho (\nabla \rho^T) \\
= -1 / \rho^2 \hat{\rho} \rho^T + 1 / \rho \left[ \hat{x} \hat{x}^T + \hat{y} \hat{y}^T \right] \\
= 1 / \rho \left[ \tilde{1} || - \hat{\rho} \rho^T \right]
\]

where we define:

\[
\rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}
\]

and \( \tilde{1} || = \hat{x} \hat{x}^T + \hat{y} \hat{y}^T \). The third part was done previously, so we just copy it from there to obtain:

\[
\nabla \nabla^T J_0(k||\rho) e^{ik_z|z-h|} = -k|| \left[ \left( -k|| J_2(k||\rho) + \frac{1}{\rho} J_1(k||\rho) \right) \hat{\rho} \pm ik_z J_1(k||\rho) \hat{z} \right] e^{ik_z|z-h|} \hat{\rho}^T
\]
\[ -k_\parallel \left[ \frac{1}{\rho} J_1(k_\parallel \rho)e^{ik_\parallel z-h} \right] \left[ \hat{1}_\parallel - \hat{p}\hat{p}^T \right] \\
\pm ik_z \left[ -k_\parallel J_1(k_\parallel \rho)\hat{p} \pm ik_z J_0(k_\parallel \rho)\hat{z} \right] e^{ik_\parallel z-h} \hat{z}^T \\
= e^{ik_\parallel z-h} \left[ (k_\parallel^2 J_2) \hat{p}\hat{p}^T + \left( \mp ik_z k_\parallel J_1 \right) \hat{z}\hat{p}^T + \left( \mp ik_z k_\parallel J_1 \right) \hat{p}\hat{z}^T + \right. \\
\left. \left( -k_\parallel^2 J_0 \right) \hat{z}\hat{z}^T + \left( -\hat{1}_\parallel /\rho \right) \hat{J}_1 \right] \hat{1}_\parallel \\
= e^{ik_\parallel z-h} \left[ (k_\parallel^2 J_2) \hat{p}\hat{p}^T + \left( \mp ik_z k_\parallel J_1 \right) \left( \hat{z}\hat{p}^T + \hat{p}\hat{z}^T \right) + \right. \\
\left. \left( -k_\parallel^2 J_0 \right) \hat{z}\hat{z}^T + \left( -k_\parallel^2 J_0 \right) \hat{1}_\parallel + \left( k_\parallel^2 J_0 - (k_\parallel /\rho) J_1 \right) \hat{1}_\parallel \right] \\
= e^{ik_\parallel z-h} \left[ (k_\parallel^2 J_2) \hat{p}\hat{p}^T + \left( \mp ik_z k_\parallel J_1 \right) \left( \hat{z}\hat{p}^T + \hat{p}\hat{z}^T \right) + \right. \\
\left. \left( -k_\parallel^2 J_0 \right) \hat{1}_\parallel + \left( k_\parallel^2 J_0 - (k_\parallel /\rho) J_1 \right) \hat{1}_\parallel \right] \\
\quad (1'33) \]

Notice that we have used the unit dyad, defined as \( \hat{1} = \hat{x}\hat{x}^T + \hat{y}\hat{y}^T + \hat{z}\hat{z}^T \). Also notice that we have introduced an extra \( \hat{1}_\parallel \) term artificially to get rid of the \( \hat{z}\hat{z}^T \) term. But it will turn out that we will need the result in its original form later.

The above result is used to evaluate the following expression:

\[ \left[ \nabla\nabla^T + k^2 \right] J_0 e^{ik_\parallel z-h} = e^{ik_\parallel z-h} \left[ k_\parallel^2 J_2 \hat{p}\hat{p}^T + ik_z k_\parallel J_1 \left( \hat{z}\hat{p}^T + \hat{p}\hat{z}^T \right) + \left( k_\parallel^2 J_0 - (k_\parallel /\rho) J_1 \right) \hat{1}_\parallel \right] \]

Now we can give the full expression for the electric field of a dipole expanded into cylindrical waves:

\[ \mathbf{E}(\mathbf{r}) = \left[ T_1 \hat{1}_\parallel + T_{pp} \hat{p}\hat{p}^T + T_{pz} \hat{z}\hat{p}^T + T_{pz} \hat{p}\hat{z}^T + T_{||} \hat{1}_\parallel \right] \mathbf{p}_j \quad (1'34) \]

\[ T_1 = \frac{i}{4\pi \varepsilon_0} \int_0^\infty dk_\parallel^3 k_\parallel^3 J_0(k_\parallel \rho)e^{ik_\parallel z-h} \quad (1'35) \]

\[ T_{pp} = \frac{i}{4\pi \varepsilon_0} \int_0^\infty dk_\parallel^3 k_\parallel^3 J_2(k_\parallel \rho)e^{ik_\parallel z-h} \quad (1'36) \]

\[ T_{pz} = T_{pz} = \pm \frac{1}{4\pi \varepsilon_0} \int_0^\infty dk_\parallel^3 k_\parallel^3 J_1(k_\parallel \rho)e^{ik_\parallel z-h} \quad (1'37) \]

\[ T_{||} = \frac{i}{4\pi \varepsilon_0} \int_0^\infty dk_\parallel^3 \left[ k_\parallel k_\parallel J_0(k_\parallel \rho) - \frac{k_\parallel^2}{k_\parallel \rho} J_1(k_\parallel \rho) \right] e^{ik_\parallel z-h} \quad (1'38) \]

The coefficients in the formula can be calculated analytically. Yet, it might be (and indeed will be) necessary to use the integral representations as well. For instance one might have some integrals resembling the ones we have here, but with very slight differences which actually make it impossible to find analytical solutions. Then the integrations should be carried out explicitly.

Some care must be taken when implementing these and such formulas into a computer program. First of all, one needs a good integrating routine, and a good Bessel function generator. ‘Good’ here means that the routines must be precise and very fast. For the beginning, the “Numerical Recipes” routines midpnt and midinf for integration, and the functions bessj0, bessj1, and bessj for generating Bessel functions are ‘good’. But these need to be modified to perform complex integration.
Furthermore, the special case of $\rho$ equal to zero must be carefully considered. Notice that in the fifth element of the transfer tensor, namely $T_\parallel$ there is a division by zero now. To avoid this, we slightly change the formula giving this component. To do this we use a recurrence relation taken from *Abramowitz & Stegun*, eq.(9.1.27):

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \quad (1'39)$$

which takes the following form for $n = 1$:

$$\frac{1}{\rho}J_1(k||\rho) = \frac{k||}{2} \left[ J_0(k||\rho) + J_2(k||\rho) \right]$$

We thus obtain a new formula for $T_\parallel$, which is superior to the previous form in that it can now handle cases of zero $\rho$, i.e. points in space along the z axis.

$$T_\parallel = \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \left[ k_\parallel k_z J_0(k||\rho) - \frac{k^3}{2k_z} \left( J_0(k||\rho) + J_2(k||\rho) \right) \right] e^{ik_z|z-h|} \quad (1'40)$$

This formula could have been of great use for us, but as we will discuss shortly, the formulas listed above simplify a great deal and we obtain much better results.

The set of formulas given above, eqs.(1’34) through (1’38) was obtained through the trick in eq.(1’33). Had we avoided this we would obtain the following second set of formulas:

$$E(r) = \left[ G_1 1 + G_{pp} \hat{p}^T + G_{ppp} (\hat{z}\hat{p}^T + \hat{p}\hat{z}^T) + G_{||} 1_|| + G_{zz} \hat{z}\hat{z}^T \right] p_j \quad (1'41)$$

$$G_1 = \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \frac{k^2}{k_z} J_0(k||\rho)e^{ik_z|z-h|} \quad (1'42)$$

$$G_{pp} = \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \frac{k^3}{k_z} J_2(k||\rho)e^{ik_z|z-h|} \quad (1'43)$$

$$G_{ppp} = \pm \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \frac{k^2}{k_z} J_1(k||\rho)e^{ik_z|z-h|} \quad (1'44)$$

$$G_{||} = \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \left( -\frac{k^2}{k_z\rho} J_1(k||\rho) \right) e^{ik_z|z-h|} \quad (1'45)$$

$$G_{zz} = \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \left( -k_\parallel k_z J_0(k||\rho) \right) e^{ik_z|z-h|} \quad (1'46)$$

In what follows we will play around with these two sets of equations. We will transform them into different, simpler integrations, and will calculate explicitly analytical solutions for the second set. We will also explore their behavior for special cases, and see how they are related to each other and to the spherical expansion. The report, in this sense, is quite extensive in handling the calculations; it includes far too much information than it is necessary for simple wave expansions. But we believe that in order to have a real grasp of the mathematics, and to suit as many diverse situations as possible, we had to do that. In this respect, the report is also a quite good source on integration of Bessel functions.

A final comment before we proceed. Notice that the second set can be transformed into the first set by adding and subtracting an extra $1_||$ term so that the $\hat{z}\hat{z}^T$ tensor vanishes (by transforming into $1$). In this way we have to deal with four tensors only. The important point is that we can still do the same thing after we have explicitly calculated the coefficients of the second set. The calculation will be done soon.
1.6.2 Matrices

Here we present the interested reader with the general form of the matrices (or dyads, or tensors) which came out of the derivation of the preceding section.

\[
\hat{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
\hat{1}_{||} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
\hat{\rho} \hat{\rho}^T = \frac{1}{x^2 + y^2} \begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
\hat{z} \hat{\rho}^T = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{pmatrix}
\]

\[
\hat{\rho} \hat{z}^T = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

1.6.3 Some Mathematical Manipulation of the Integrals

To ease the numerical implementation of the integrals it would be highly desirable to remove singularities caused by \( k_z \) at \( k_{||} = k \). Using integration by parts we might be able to achieve our goal. Here we will play around with the previously derived two sets of formulas for the transfer tensor, eqs.(1’34) and (1’41).

A. First set of equations, eq.(1’34)

In this section we will try to simplify the integrations of the first set of expansions without losing their cylindrical behaviour. At the end we will also show a way of calculating analytically the coefficients.

We begin with the Sommerfeld identity:

\[
\frac{e^{ikr}}{r} = i \int_0^\infty dk_{||} \frac{k_{||}}{k_z} J_0(k_{||}\rho) e^{ik_z|z-h|} \]

\[
= -i \int_0^\infty dk_{||} \frac{dk_z}{dk_{||}} J_0(k_{||}\rho) e^{ik_z|z-h|} \]

\[
= -\frac{1}{|z-h|} \int_0^\infty dk_{||} J_0(k_{||}\rho) \frac{d}{dk_{||}} e^{ik_z|z-h|} \]

\[
= -\frac{1}{|z-h|} \left( J_0(k_{||}\rho) e^{ik_z|z-h|}\bigg|_0^\infty - \int_0^\infty dk_{||} |e^{ik_z|z-h|}| dJ_0(k_{||}\rho) \right) \]

\[
= -\frac{1}{|z-h|} \left( 0 - J_0(0) e^{ik|z-h|} + \int_0^\infty dk_{||} |e^{ik|z-h|}| \rho J_1(k_{||}\rho) \right) \]
than the previous when implemented into a computer program.

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The advantage of this new formula is that it gives about two times more precise results than the previous when implemented into a computer program.

What happens when $z = h$? Apparently, our new formula is poorer in handling this special case since there is a zero in the denominator. Fortunately there is a zero in the numerator as well. This follows directly from eq. (1'72), which will be described later. This leaves a possibility of applying the L'Hôpital rule.

We now proceed with the other integrals:

$$T_1 = \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel k_\parallel^3 J_0(k_\parallel \rho) e^{ik_\parallel z - h}$$

$$= \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel k_\parallel^3 \frac{dk_\parallel}{dk_\parallel} J_0(k_\parallel \rho) e^{ik_\parallel z - h}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel k_\parallel^2 J_0(k_\parallel \rho) \frac{d}{dk_\parallel} e^{ik_\parallel z - h}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \left[ k_\parallel^2 J_0(k_\parallel \rho) e^{ik_\parallel z - h} \right]_0^\infty - \int_0^\infty dk_\parallel e^{ik_\parallel z - h} \frac{d}{dk_\parallel} k_\parallel^2 J_0(k_\parallel \rho)$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \left[ 0 - \int_0^\infty dk_\parallel e^{ik_\parallel z - h} \left( 2k_\parallel J_0(k_\parallel \rho) + k_\parallel^2 \frac{d}{dk_\parallel} J_0(k_\parallel \rho) \right) \right]$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \left( 2k_\parallel J_0(k_\parallel \rho) - k_\parallel^2 \rho J_1(k_\parallel \rho) \right) e^{ik_\parallel z - h}$$

(1'48)

For $z = h$ we obtain a zero over zero case, as can be seen from eq. (1'73) to be discussed, indicating that we might again try using the L'Hôpital rule.

$$T_{\rho \rho} = \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel k_\parallel^3 J_2 e^{ik_\parallel z - h}$$

$$= \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel k_\parallel^3 \frac{dk_\parallel}{dk_\parallel} J_2 e^{ik_\parallel z - h}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel k_\parallel^2 J_2 \frac{d}{dk_\parallel} e^{ik_\parallel z - h}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \left[ k_\parallel^2 J_2 e^{ik_\parallel z - h} \right]_0^\infty - \int_0^\infty dk_\parallel e^{ik_\parallel z - h} \frac{d}{dk_\parallel} k_\parallel^2 J_2$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel e^{ik_\parallel z - h} \left[ 2k_\parallel J_2 + k_\parallel^2 \frac{d}{dk_\parallel} J_2 \right]$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \left[ 2k_\parallel J_2 + k_\parallel^2 \left( -\rho J_3 + \frac{2}{k_\parallel} J_2 \right) \right] e^{ik_\parallel z - h}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \left[ 4k_\parallel J_2 - k_\parallel^2 \rho J_3 \right] e^{ik_\parallel z - h}$$

We don’t let the integral go at this point. Rather we consult eq. (1'39), which has the following form for $n = 2$:

$$4J_2 = k_\parallel \rho \left[ J_1 + J_3 \right]$$
Then:

\[
T_{\rho\rho} = \frac{1}{4\pi\epsilon_0 |z - h|} \int_0^\infty dk_\parallel \left[ k_\parallel^2 \rho (J_1 + J_3) - k_\parallel^2 \rho J_3 \right] e^{ik_\parallel z - h} \\
= \frac{1}{4\pi\epsilon_0 |z - h|} \int_0^\infty dk_\parallel k_\parallel^2 \rho J_1 e^{ik_\parallel z - h}
\]

Finally:

\[
T_\parallel = \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel k_\parallel k_\perp J_0 e^{ik_\parallel z - h} \\
= \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel k_\parallel J_0 + \frac{k_\parallel}{\rho} \frac{d}{dk_\parallel} J_1 e^{ik_\parallel z - h} \\
= \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel k_\parallel J_0 e^{ik_\parallel z - h} + \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \frac{k_\parallel}{\rho} J_1 \frac{d}{dk_\parallel} e^{ik_\parallel z - h}
\]

\[
T_\parallel - \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel k_\parallel k_\perp J_0 e^{ik_\parallel z - h} = \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \frac{k_\parallel}{\rho} J_1 \frac{d}{dk_\parallel} e^{ik_\parallel z - h} \\
= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \frac{k_\parallel}{\rho} J_1 e^{ik_\parallel z - h} \left[ \frac{1}{\rho} J_1 + \frac{k_\parallel}{\rho} \frac{d}{dk_\parallel} J_1 \right] \\
= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel e^{ik_\parallel z - h} \left[ \frac{1}{\rho} J_1 + \frac{k_\parallel}{\rho} \left( -\rho J_2 + \frac{1}{k_\parallel} J_1 \right) \right] \\
= \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel e^{ik_\parallel z - h} \left[ \frac{1}{\rho} J_1 + \frac{k_\parallel}{\rho} \left( -\rho J_2 + \frac{1}{k_\parallel} J_1 \right) \right]
\]

Now we go back to eq.(1’39). Then:

\[
T_\parallel - \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel k_\parallel k_\perp J_0 e^{ik_\parallel z - h} = -\frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel e^{ik_\parallel z - h} \left[ k_\parallel (J_0 + J_2) - k_\parallel J_2 \right] \\
= -\frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel J_0 e^{ik_\parallel z - h}
\]

\[
T_\parallel = \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \left( k_\parallel + \frac{i}{|z - h|} \right) J_0 e^{ik_\parallel z - h}
\]

How many integrals need to be calculated in all? Not more than several. For clarity, we list our last results below.

**Results**

Define:

\[
\Psi_\nu = k_\parallel^{\nu+1} J_\nu (k_\parallel |\rho|) e^{ik_\parallel z - h}
\]

\[
\chi_\nu = \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel \Psi_\nu
\]
Then:

\[
T_1 = \frac{2}{|z - h|} \chi_0 - \frac{\rho}{|z - h|} \chi_1 \quad (1'53)
\]

\[
T_{\rho \rho} = \frac{\rho}{|z - h|} \chi_1 \quad (1'54)
\]

\[
T_{z \rho} = T_{\rho z} = \pm \chi_1 \quad (1'55)
\]

\[
T_{\|} = \frac{i}{4\pi \varepsilon_0} \int_0^{\infty} dk_{\|} k_z \Psi_0 - \frac{1}{|z - h|} \chi_0 \quad (1'56)
\]

We can further transform these formulas into better looking mathematical expressions. First notice that the following three equations hold:

\[
\Psi_1 = -\frac{d}{d\rho} \Psi_0 \quad (1'57)
\]

\[
\chi_1 = -\frac{d}{d\rho} \chi_0 \quad (1'58)
\]

\[
\frac{i}{4\pi \varepsilon_0} \int_0^{\infty} dk_{\|} k_z \Psi_0 = \frac{d}{d|z - h|} \chi_0 \quad (1'59)
\]

so:

\[
T_1 = \frac{1}{|z - h|} \left( 2 + \rho \frac{d}{d\rho} \right) \chi_0 \quad (1'60)
\]

\[
T_{\rho \rho} = \frac{1}{|z - h|} \left( \rho \frac{d}{d\rho} \right) \chi_0 \quad (1'61)
\]

\[
T_{z \rho} = T_{\rho z} = \mp \left( \rho \frac{d}{d\rho} \right) \chi_0 \quad (1'62)
\]

\[
T_{\|} = \left( \frac{d}{d|z - h|} - \frac{1}{|z - h|} \right) \chi_0 \quad (1'63)
\]

Now we see that if the analytical evaluation of \( \chi_0 \) was possible then we could calculate all the required coefficients exactly. It turns out that this evaluation is indeed possible.

\[
\chi_0 = \frac{1}{4\pi \varepsilon_0} \int_0^{\infty} dk_{\|} k_{\|} J_0(k_{\|} \rho) e^{ik_{\|} |z - h|}
\]

\[
= \pm \frac{1}{4\pi \varepsilon_0} \int_0^{\infty} dk_{\|} k_{\|} J_0(k_{\|} \rho) \frac{d}{dz} e^{ik_{\|} |z - h|}
\]

\[
= \mp \frac{1}{4\pi \varepsilon_0} \frac{d}{dz} \left( i \int_0^{\infty} dk_{\|} \frac{k_{\|}}{k_z} J_0(k_{\|} \rho) e^{ik_{\|} |z - h|} \right)
\]

\[
= \mp \frac{1}{4\pi \varepsilon_0} \frac{d}{dz} \left( \frac{e^{ikr}}{r} \right)
\]

\[
= \mp \frac{1}{4\pi \varepsilon_0} \frac{z - h}{r} \left( ik - \frac{1}{r} \right) e^{ikr} \quad (1'64)
\]

The rest, i.e. the explicit calculation of the coefficients is trivial and is left as an exercise for the interested reader.
B. Second set of equations, eq.(1′41)

Here we will derive analytical expressions for the integrations in the second set of the cylindrical wave expansion of the electric field.

Using the Sommerfeld expression, eq.(1′30) we obtain:

\[ G_1 = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk_\parallel \frac{k_\parallel^4}{k_z} J_0(k_\parallel\rho) e^{ik_\parallel z-h} \]
\[ = \frac{k_\parallel^2}{4\pi\varepsilon_0} \left( i \int_0^\infty dk_\parallel \frac{k_\parallel}{k_z} J_0(k_\parallel\rho) e^{ik_\parallel z-h} \right) \]
\[ = \frac{k_\parallel^2}{4\pi\varepsilon_0} \frac{e^{ikr}}{r} \]  \hspace{1cm} (1′65)

Making use of eq.(1′32) as well, we obtain:

\[ G_\parallel = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk_\parallel \left( -\frac{k_\parallel^2}{k_\parallel^2} J_1(k_\parallel\rho) \right) e^{ik_\parallel z-h} \]
\[ = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk_\parallel \left( \frac{k_\parallel}{k_\parallel^2} J_0(k_\parallel\rho) \right) e^{ik_\parallel z-h} \]
\[ = \frac{1}{4\pi\varepsilon_0} \left[ i \int_0^\infty dk_\parallel \frac{k_\parallel}{k_z} J_0(k_\parallel\rho) e^{ik_\parallel z-h} \right] \]
\[ = \frac{1}{4\pi\varepsilon_0} \left[ i \left( \frac{e^{ikr}}{r} \right) \right] \]
\[ = \frac{1}{4\pi\varepsilon_0} \left[ i \left( \frac{e^{ikr}}{r} \right) \right] \]
\[ = \frac{1}{4\pi\varepsilon_0} \left( \frac{ik}{r} - \frac{1}{r^2} \right) e^{ikr} \]  \hspace{1cm} (1′66)

where we have used the definitions of \( \rho \) and of \( r \):

\[ r = \sqrt{\rho^2 + (z-h)^2} \]

\[ G_{\rho\rho} = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk_\parallel \frac{k_\parallel^3}{k_z} J_2(k_\parallel\rho) e^{ik_\parallel z-h} \]
\[ = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk_\parallel \frac{k_\parallel^2}{k_z} \left( \frac{1}{\rho} - \frac{d}{d\rho} \right) J_1(k_\parallel\rho) e^{ik_\parallel z-h} \]
\[ = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk_\parallel \frac{k_\parallel}{k_z} \left( \frac{1}{\rho} + \frac{d^2}{d\rho^2} \right) J_0(k_\parallel\rho) e^{ik_\parallel z-h} \]
\[ = \frac{1}{4\pi\varepsilon_0} \left( \frac{d^2}{d\rho^2} - \frac{1}{\rho} \frac{d}{d\rho} \right) \frac{e^{ikr}}{r} \]
\[ = \frac{1}{4\pi\varepsilon_0} \left[ \left( \frac{d}{d\rho} - \frac{1}{\rho} \right) \frac{d}{d\rho} \right] \frac{e^{ikr}}{r} \]
\[ = \frac{1}{4\pi\varepsilon_0} \left[ \left( \frac{d}{d\rho} - \frac{1}{\rho} \right) \frac{d}{d\rho} \right] \frac{e^{ikr}}{r} \]
\[ = \frac{1}{4\pi\varepsilon_0} \left( \frac{d}{d\rho} - \frac{1}{\rho} \right) \left[ \left( \frac{ik e^{ikr}}{r} - \frac{e^{ikr}}{r^2} \right) \right] \frac{dr}{d\rho} \]
Fields Generated by an Oscillating Dipole

\[ G_{\rho z} = \pm \frac{1}{4\pi\varepsilon_0} \int_0^\infty dk|k||J_1(k||\rho)\rho e^{ik|z-h|} \]
\[ = -\frac{i}{4\pi\varepsilon_0} \int_0^\infty dk|k|^2 J_1(k||\rho) \frac{d}{dz} e^{ik|z-h|} \]
\[ = \rho \frac{d}{dz} \left[ \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk|k| \left( -\frac{k^2}{k^2\rho} J_1(k||\rho) \right) e^{ik|z-h|} \right] \]
\[ = \rho \frac{d}{dz} [G_{||}] \]
\[ = \frac{1}{4\pi\varepsilon_0} \left[ \left( -k^2 - \frac{3ik}{r} + \frac{3}{r^2} \right) \rho(z-h) \right] \frac{e^{ikr}}{r} \quad (1'67) \]

\[ G_{zz} = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk|k||J_0(k||\rho)\rho e^{ik|z-h|} \]
\[ = \frac{1}{4\pi\varepsilon_0} \frac{d^2}{dz^2} \left( i \int_0^\infty dk|k|^2 J_0(k||\rho) e^{ik|z-h|} \right) \]
\[ = \frac{1}{4\pi\varepsilon_0} \frac{d^2}{dz^2} \left( \frac{e^{ikr}}{r} \right) \]
\[ = \frac{1}{4\pi\varepsilon_0} \left[ \left( -k^2 - \frac{3ik}{r} + \frac{3}{r^2} \right) \frac{(z-h)^2}{r^2} + \left( \frac{ik}{r} - \frac{1}{r^2} \right) \right] \frac{e^{ikr}}{r} \quad (1'68) \]

Using the analytical results for this second set, we can jump on to the first set through appropriate transformations (see eq.(1'33)).

1.6.4 Special Cases

Here we discuss some special cases both before and after taking the gradient of the Sommerfeld identity. This part is not related to anything that will follow; the uninterested reader might want to skip it altogether...

It is important to realize that we cannot first substitute a value (like \( z = 0 \)) into the Sommerfeld expression and then take its gradient. In other words, the electric field formula eq.(1'31) is valid for the 3-D space, not for a 2-D plane; and we cannot compress it into a plane (by letting \( z = h \) for instance) and then take the gradient to obtain the electric field for that plane. What we can do is to first apply the gradient (and thus obtain the general solution) and only then collapse the result into a plane, as appropriate.

A. Special cases and validity of the Sommerfeld expression

It is possible to check the validity of the Sommerfeld expression numerically. But it is possibly wiser to check the correctness of the program itself by calculating this expression, because
it is true (we state this without proof, but not without justification, as we will shortly see). Here we rewrite the expression in question:

$$\frac{e^{ikr}}{r} = i \int_0^\infty dk k J_0(k \rho) e^{ik |z - h|}$$

where

$$k_z = \sqrt{k^2 - k_{\parallel}^2}$$

$$r = \sqrt{\rho^2 + (z - h)^2}$$

We will use the Sommerfeld expression in deriving important formulas for future reference. The derivation of the expression itself is beyond the scope of this report.

A.1 Special case $k = 0$

When $k$ is equal to zero, $k_z$ becomes purely imaginary. But it is still not trivial to say that $k_z = ik_{\parallel}$, because an imaginary number has two roots. Physics rather than mathematics forces us to choose the above form for $k_z$; otherwise the exponential term in the integral of the Sommerfeld expression will explode. So we proceed:

$$k_z = ik_{\parallel}$$

$$\frac{e^{ikr}}{r} = \frac{1}{r} = i \int_0^\infty dk \frac{k_{\parallel}}{ik_{\parallel}} J_0(k_{\parallel} \rho) e^{-k_{\parallel} |z - h|}$$

$$\frac{1}{r} = \int_0^\infty dk J_0 e^{-k_{\parallel} |z - h|}$$

$$\frac{1}{\sqrt{\rho^2 + (z - h)^2}} = \int_0^\infty dk J_0(k_{\parallel} \rho) e^{-k_{\parallel} |z - h|}$$

The resulting formula was checked numerically, thus establishing its reliability beyond any doubt. The formula can be rewritten in the following form:

$$\frac{1}{\sqrt{1 + z^2}} = \int_0^\infty dt J_0(t) e^{-zt} \tag{1’70}$$

A.2 Special case $\rho = 0$

$$r = |z - h|$$

$$\frac{e^{ikr}}{r} = i \int_0^\infty dk \frac{k_{\parallel}}{k_z} J_0(0) e^{ik_z r}$$

$$= -i \int_0^\infty dk \frac{dk_z}{dk_{\parallel}} e^{ik_z r}$$

$$= -\frac{e^{ik_z r}}{r} \bigg|_{k_{\parallel} = 0}$$

$$= -\frac{e^{-\infty}}{r} + \frac{e^{ikr}}{r}$$

$$= \frac{e^{ikr}}{r}$$

Q.E.D.
A.3 Special case $\rho = 0$ and $k = 0$
This case is trivial and is left as an exercise for the interested reader.

A.4 Special case $z = h$

$$
\frac{r}{\rho} = e^{ikr} = e^{ik\rho} = i \int_0^\infty dk_z \frac{k_z J_0(k_z\rho)}{k_z} \quad (1’71)
$$

We assume the reliability of this expression.

A.5 Special case $z = h$ and $k = 0$

$$
k_z = ik ||
\frac{1}{\rho} = \int_0^\infty dk_z J_0(k_z\rho)
1 = \int_0^\infty dt J_0(t)
$$

The above equation is true. See Abramowitz & Stegun, eq.(11.4.17), namely:

$$
\int_0^\infty J_\nu(t) dt = 1 \quad (1’72)
$$

At this point we would like to quote another important equation from the same source, eq.(11.4.16):

$$
\int_0^\infty t^\mu J_\nu(t) dt = \frac{2^\mu \Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} \quad (1’73)
$$

where $\Gamma$ is the famous Gamma function defined by eq.(6.1.1):

$$
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} dt \quad (1’74)
$$

The book also gives listings of the function evaluated at various points.

B. Special cases for the electric field

The general derivation of the electric field was discussed above. Here we discuss the electric field expressions for two important special cases: the case of $\rho = 0$ and that of $z = h$. Before proceeding, we should point out one more time that in order to examine special cases for the electric field (like the ones above) we first need a general formulation (like the two sets of equations we obtained) and only then can we put in special values. In other words, in general it is not true that by beginning from a special form of the Sommerfeld identity, we can derive the electric field for that special case.
B.1 Special case $\rho = 0$

This case is quite straightforward if one knows the behaviour of the Bessel functions. The first set of equations simply becomes:

$$T_1 = i \frac{k^2}{k_z} \frac{e^{ik_z|z-h|}}{\rho}$$

$$T_{pp} = T_{pz} = T_{pz} = 0$$

$$T_\parallel = i \frac{k^2}{k_z} \frac{e^{ik_z|z-h|}}{\rho}$$

And the second:

$$G_1 = i \frac{k^2}{k_z} \frac{e^{ik_z|z-h|}}{\rho}$$

$$G_{pp} = G_{pz} = 0$$

$$G_\parallel = -i \frac{k^2}{k_z} \frac{e^{ik_z|z-h|}}{\rho}$$

$$G_{zz} = -i \frac{k^2}{k_z} \frac{e^{ik_z|z-h|}}{\rho}$$

Recalling the definitions of $\hat{1}$, $\hat{1}_\parallel$, and $\hat{z}$ we see that these two sets are identical. It is also possible to show that the second set is equivalent to the spherical expansion, but this is left as an exercise for the reader.

B.2 Special case $z = h$

We shall investigate the second set of equations only. The rest will be left as an exercise for the reader. (Recall that the first set was problematic on this plane. We had argued while we were simplifying the integrations that it might be necessary to employ the L'Hôpital rule.)

For this case we simply have $r = \rho$. So our equations become:

$$G_1 = \frac{k^2}{4\pi\epsilon_0} \frac{e^{ik\rho}}{\rho}$$

$$G_\parallel = \frac{1}{4\pi\epsilon_0} \left( \frac{ik}{\rho} - \frac{1}{\rho^2} \right) \frac{e^{ik\rho}}{\rho}$$

$$G_{\rho\rho} = \frac{1}{4\pi\epsilon_0} \left( -k^2 - \frac{3ik}{\rho} + \frac{3}{\rho^2} \right) \frac{e^{ik\rho}}{\rho}$$

$$G_{\rho\rho} = 0$$

$$G_{zz} = -\frac{1}{4\pi\epsilon_0} \left( \frac{ik}{\rho} - \frac{1}{\rho^2} \right) \frac{e^{ik\rho}}{\rho}$$
Notice that these are equivalent to the spherical expansion.

Recall further that we had refrained from the division by $\rho$ in the $T|$ term in the general formulas, in order to avoid a division by zero for the case of $\rho = 0$. Apparently we need not bother about such divisions now since it is highly unlikely that we try calculating the electric field for $\rho = 0$ and $z = \frac{h}{2}$ at the same time. This would simply mean that we are calculating the field at the position of the dipole.

1.7 Plane Wave Expansion

We use the Weyl expansion of a spherical wave into plane waves, which has the following form:

$$
e^{ikr} = \frac{i}{2\pi} \int_{-\infty}^{\infty} dk_{||} e^{i(k_{||}\rho + k_{z}|z|)} k_{z}$$

where:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\rho = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \quad k_{||} = \begin{pmatrix} k_{x} \\ k_{y} \\ 0 \end{pmatrix} \quad \text{and} \quad k_{z} = \sqrt{k^2 - k_{||}^2} = \sqrt{k^2 - k_{x}^2 - k_{y}^2}$$

Notice that now we have a two dimensional integration over the entire $k_{x} - k_{y}$ plane. Also notice that we have implicitly assumed that the dipole is on the $x - y$ plane. If this is not the case then simply replace $z$ by $z - \frac{h}{2}$ as was previously done in the cylindrical wave expansion.

A final, but very important remark before we proceed: $k_{z}$ as defined here, is always positive, or at least zero (we consider only real values for the time). So, negative $z$ components of wave vectors will be denoted by $-k_{z}$. This definition shall be used throughout the text that follows. (Actually, you can think of $k_{z}$ as an absolute value.)

The calculation of the transfer tensor is straightforward:

$$E(r) = T(r)p$$

$$T(r) = \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} dk_{||} \left[ \nabla\nabla^T + k^2 \right] e^{i(k_{||}\rho + k_{z}|z|)} \frac{e^{ikr}}{k_{z}}$$

(1'76)

Since there is an absolute value in the exponential we need to distinguish between the upper and lower half planes. For the upper half plane we have:

$$T(r) = \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} dk_{||} \left[ \nabla\nabla^T + k^2 \right] e^{i(k_{||}\rho + k_{z}|z|)} \frac{e^{ikr}}{k_{z}}$$

$$= \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} dk_{||} \left[ k^2 - kk^T \right] e^{i(k_{||}\rho + k_{z}z)} \frac{e^{i(k_{||}\rho + k_{z}z)}}{k_{z}}$$

$$k = \begin{pmatrix} k_{x} \\ k_{y} \\ k_{z} \end{pmatrix}$$
Notice that:

\[
kk^T = \begin{pmatrix} k_x k_x & k_x k_y & k_x k_z \\ k_y k_x & k_y k_y & k_y k_z \\ k_z k_x & k_z k_y & k_z k_z \end{pmatrix}
\]

Similarly for the lower half plane:

\[
T(r) = \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} d|k| \left| k^2 - kk^T \right| e^{i(|k| r - k_z z)} \frac{1}{k_z} \\
= \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} d|k| \left| k^T k - kk^T \right| e^{i(|k| r + k_z z|\text{sign}(z)|)} \frac{1}{k_z}
\]

\[k = \begin{pmatrix} k_x \\ k_y \\ -k_z \end{pmatrix}
\]

\[kk^T = \begin{pmatrix} k_x k_x & k_x k_y & -k_x k_z \\ k_y k_x & k_y k_y & -k_y k_z \\ -k_z k_x & -k_z k_y & k_z k_z \end{pmatrix}
\]

We can combine the two formulas with the help of the sign function:

\[
T(r) = \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} d|k| |\mathcal{M}| e^{i(|k| r + k_z z|\text{sign}(z)|)} |k_z|
\]

\[\mathcal{M} = \begin{pmatrix} k^2 - k_x k_x & -k_x k_y & -k_x k_z \\ -k_y k_x & k^2 - k_y k_y & -k_y k_z \\ -k_z k_x & -k_z k_y & k^2 - k_z k_z \end{pmatrix} \]

The product \(|\text{sign}(z)|\) is simply \(|z|\).

As can be seen from the resulting formula, one needs to calculate six independent elements only. But do we have to calculate these integrals? Believe me, it takes so much time that it is quite questionable whether the effort is worth it. Let’s play with the formula to see how are things getting...

\[
T_{xx}(r) = \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} d|k| \left( k^2 - k_x k_x \right) e^{i(|k| r + k_z z|\text{sign}(z)|)} \frac{1}{k_z}
\]

\[
= \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} d|k| \left( k^2 + \frac{d^2}{dx^2} \right) e^{i(|k| r + k_z z|\text{sign}(z)|)} \frac{1}{k_z}
\]

\[
= \frac{1}{4\pi^2\epsilon_0} \left( k^2 + \frac{d^2}{dx^2} \right) \frac{e^{ikr}}{r}
\]

\[
= \frac{1}{4\pi^2\epsilon_0} \left[ \left( k^2 + \frac{ik}{r} - \frac{1}{2} \right) - \left( k^2 + \frac{3ik}{r} - \frac{3}{2} \right) \frac{x^2}{r^2} \right] e^{ikr}
\]

The next diagonal term is obvious:

\[
T_{yy} = \frac{1}{4\pi^2\epsilon_0} \left[ \left( k^2 + \frac{ik}{r} - \frac{1}{2} \right) - \left( k^2 + \frac{3ik}{r} - \frac{3}{2} \right) \frac{y^2}{r^2} \right] e^{ikr}
\]
But the last diagonal term needs some attention:

\[
T_{zz} = \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} dk_z \left( k^2 - k_z k_z \right) \frac{e^{i(k_z\rho + k_z z)}}{k_z} \\
= \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} dk_z \left( k^2 + \frac{1}{\text{sign}(z)} \frac{d^2}{dz^2} \right) \frac{e^{i(k_z\rho + k_z z \text{sign}(z))}}{k_z} \\
= \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} dk_z \left( k^2 + \frac{d^2}{dz^2} \right) \frac{e^{i(k_z\rho + k_z z \text{sign}(z))}}{k_z} \\
= \frac{1}{4\pi\epsilon_0} \left( k^2 + \frac{ik}{r} - \frac{1 + k^2 z^2}{r^2} - \frac{3ikz^2}{r^3} + \frac{3z^2}{r^4} \right) e^{ikr} \\
= \frac{1}{4\pi\epsilon_0} \left[ \left( k^2 + \frac{ik}{r} - \frac{1}{r^2} \right) - \left( k^2 + \frac{3ikz^2}{r^2} - \frac{3z^2}{r^4} \right) \frac{e^{ikr}}{r} \right] (1'80)
\]

In view of what has been already done, the rest is almost trivial:

\[
T_{xy} = T_{yx} = \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} dk_y \left( -k_x k_y \right) \frac{e^{i(k_y\rho + k_y z)}}{k_y} \\
= \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} dk_y \left( \frac{d^2}{dxdy} \right) \frac{e^{i(k_y\rho + k_y z)}}{k_y} \\
= \frac{1}{4\pi\epsilon_0} \left( \frac{d^2}{dxdy} \right) \frac{e^{ikr}}{r} \\
= \frac{1}{4\pi\epsilon_0} \left( \frac{d}{dy} \left( \frac{d e^{ikr}}{dr} \right) \right) \\
= \frac{1}{4\pi\epsilon_0} \frac{d}{dx} \left( \frac{d e^{ikr}}{dy} \right) \\
= \frac{1}{4\pi\epsilon_0} \frac{d}{dx} \left( ik - \frac{1}{r} \right) \frac{e^{ikr} dr}{r} \\
= \frac{1}{4\pi\epsilon_0} \left[ \left( k^2 + \frac{3ik}{r} - \frac{3}{r^2} \right) \frac{xy}{r^2} \right] e^{ikr} (1'81)
\]

\[
T_{zz} = T_{zz} = \frac{1}{4\pi\epsilon_0} \left[ \left( k^2 + \frac{3ik}{r} - \frac{3}{r^2} \right) \frac{xz}{r^2} \right] e^{ikr} (1'82)
\]

\[
T_{yz} = T_{zy} = \frac{1}{4\pi\epsilon_0} \left[ \left( k^2 + \frac{3ik}{r} - \frac{3}{r^2} \right) \frac{yz}{r^2} \right] e^{ikr} (1'83)
\]

1.8 Sommerfeld vs Weyl

The Sommerfeld and Weyl expansions are equivalent. That is, the electric fields calculated through the two expansions are the same. This can be checked numerically (which is probably the best thing to do), but in this section we will try to give a rough justification (even if not a whole proof).
Let’s first recall the two expansions:

\[
\frac{e^{ikr}}{r} = i \int_{0}^{\infty} dk ||k|| \frac{k_z}{k} J_0(k_z) e^{ik_z|z|}
\]

\[
\frac{e^{ikr}}{r} = i \frac{1}{2\pi} \int_{-\infty}^{\infty} dk ||k|| \frac{k_z}{k} e^{ik_z|\rho| + k_z|z|} e^{ik_z|z|}
\]

We begin with rewriting the Weyl expansion in cylindrical coordinates:

\[
\frac{e^{ikr}}{r} = i \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} d\theta dk ||k|| \frac{k_z}{k} e^{ik_z|\rho\cos \theta + k_z|z|} e^{ik_z|z|}
\]

Notice that the angle \(\theta\) is being measured from the vector \(\rho\). If we now compare this result with the Sommerfeld expression we see that the following equality should hold:

\[
J_0(k_z|\rho|) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{ik_z|\rho\cos \theta} d\theta
\]

We shall now prove that this equation is true. But to accomplish this tedious task we make use of Abramowitz & Stegun, and assuming their validity we quote some of the relevant equations given there. For the modified Bessel function of the first kind of order zero, which is denoted by \(I_0(z)\) we have eq.(9.6.16) which reads:

\[
I_0(z) = \frac{1}{\pi} \int_{0}^{\pi} e^{\pm iz\cos \theta} d\theta
\]

(1’84)

It follows from the general eq.(9.6.3) that the relation between this function and the usual Bessel function, \(J_0\) is given through:

\[
I_0(z) = J_0(iz)
\]

(1’85)

which implies that:

\[
I_0(iz) = J_0(-z) = J_0(z)
\]

The second part of the equality is true since Bessel functions of the first kind of even order are all even functions. This follows from the definition of these functions, given by eq.(9.1.10). Then:

\[
J_0(z) = I_0(iz) = \frac{1}{\pi} \int_{0}^{\pi} e^{\pm iz\cos \theta} d\theta
\]

\[
= \frac{1}{2\pi} \left( \int_{0}^{\pi} e^{iz\cos \theta} d\theta + \int_{0}^{\pi} e^{-iz\cos \theta} d\theta \right)
\]

\[
= \frac{1}{2\pi} \left( \int_{0}^{\pi} e^{iz\cos \theta} d\theta + \int_{0}^{\pi} e^{iz\cos(\theta + \pi)} d\theta \right)
\]

\[
= \frac{1}{2\pi} \left( \int_{0}^{\pi} e^{iz\cos \theta} d\theta + \int_{0}^{2\pi} e^{iz\cos \theta} d\theta \right)
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} e^{iz\cos \theta} d\theta
\]

(1’86)

\(^4\)We provide the reader with plenty of Bessel functions in the appendix.
Thus the Sommerfeld and Weyl expansions are mathematically equivalent. To show, however, that the same holds for the electric field expressions as written in these two expansions, i.e. that eqs.(1’34) and (1’77) are also equivalent is not trivial (and therefore is not left as an exercise to the reader). Still, it is possible to check them numerically, and see that this indeed is the case. (This is what we can call experimental mathematics.) Equivalence of the two electric fields requires that we have:

\[\begin{align*}
k_{\parallel}^2 J_0(k_{\parallel}\rho) \hat{\mathbf{1}} + k_{\parallel}^2 J_2(k_{\parallel}\rho) \hat{\mathbf{p}}^T - i \text{sign}(z) k_{\parallel} k_z J_1(k_{\parallel}\rho) \left( \hat{\mathbf{2}} \hat{\mathbf{p}}^T + \hat{\mathbf{p}} \hat{\mathbf{z}}^T \right) \\
+ \left( k_z^2 J_0(k_{\parallel}\rho) - \frac{k_{\parallel}^2}{\rho} J_1(k_{\parallel}\rho) \right) \hat{1}_{\parallel} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{M} e^{ik_{\parallel}\rho} d\theta \tag{1’87}
\end{align*}\]

where the tensor \( \mathcal{M} \) is now rewritten in cylindrical coordinates as:

\[\mathcal{M}(k_{\parallel}, \theta) = \begin{pmatrix}
k^2 - k_{\parallel}^2 \cos^2 \theta & -k_{\parallel}^2 \cos \theta \sin \theta & -k_{\parallel} k_z \cos \theta \text{sign}(z) \\
-k_{\parallel}^2 \cos \theta \sin \theta & k^2 - k_{\parallel}^2 \sin^2 \theta & -k_{\parallel} k_z \sin \theta \text{sign}(z) \\
-k_{\parallel} k_z \cos \theta \text{sign}(z) & -k_{\parallel} k_z \sin \theta \text{sign}(z) & k_{\parallel}^2
\end{pmatrix} \tag{1’88}\]

The freedom of choice of the angle is now gone, and we have to write the dot product in the exponential as:

\[k_{\parallel}\rho = k_{\parallel} x \cos \theta + k_{\parallel} y \sin \theta \tag{1’89}\]
Chapter 2

The Dipole-Surface Case I: Plane Waves and the Weyl Expansion

2.1 Boundary Conditions

In this section we will discuss the boundary conditions at the interface between two media. We will derive reflection and transmission coefficients using matrix algebra. We will then consider the electric fields on both sides of the boundary.

We shall first quickly review Maxwell’s equations, which read:

\[ \nabla \cdot E = \frac{\rho}{\epsilon} \]
\[ \nabla \cdot B = 0 \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]
\[ \nabla \times B = \mu \sigma E + \mu \epsilon \frac{\partial E}{\partial t} \]  

(2.1)

where we can make the following substitutions:

\[ D = \epsilon E \]
\[ H = \frac{1}{\mu} B \]
\[ J = \sigma E \]

Maxwell’s equations demand that the tangential components of \( E \) and \( H \) be continuous across an interface. For non-magnetic medium continuity of \( H \) implies continuity of \( B \).

Assume that there are two media separated by a boundary at the \( x - y \) plane. Also assume that the media are non-magnetic (the general assumption in classical optics). We can write the following equation for the electric field (the subscripts \( i, r \) and \( t \) will refer to the incident, reflected and transmitted/refracted waves respectively):

\[ (E_i)_{||} + (E_r)_{||} = (E_t)_{||} \]
\[ (E_0 e^{ikr})_{||} + (\mathcal{R} E_0 e^{ikr})_{||} = (\mathcal{T} E_0 e^{ikr})_{||} \]  

(2.2)

where \( \mathcal{R} \) and \( \mathcal{T} \) are 3 \times 3 reflection and transmission tensors. It is important to notice that since the incident wave vector should have a negative \( z \) component (otherwise it would not
The electric field has three components in free space. But with respect to the plane of incidence it has only two: one perpendicular to the plane \((E_s)\), and one parallel to it \((E_p)\). (It is common practice in the standard literature to point out that “p” stands for “parallel”, and “s” for “senkrecht”, which is German for perpendicular; but we will not do that!) We can write the electric field in terms of these components as:

\[
E_0 = E_s \hat{s} + E_p \hat{p}
\]

We would like to treat these two components at the same time using matrix algebra, instead of considering them separately as is the usual case in the standard literature...

### 2.1.1 The Tensor Treatment of Reflection and Transmission Phenomena for a Particular Orientation

At this point for simplicity we make one assumption more: we assume that the \(x\) direction is perpendicular to the plane of incidence. Notice that this implies that we have:

\[
k_i = \begin{pmatrix} 0 \\ k_{iy} \\ -k_{iz} \end{pmatrix}, \quad k_r = \begin{pmatrix} 0 \\ k_{iy} \\ 0 \end{pmatrix}, \quad \text{and} \quad k_t = \begin{pmatrix} 0 \\ k_{iy} \\ -k_{tz} \end{pmatrix}
\]

To avoid confusion, we have used \(k_i\) and \(k_r\) to denote the wave vectors \(k\) and \(k\), respectively. We shall stick to this notation from now on. Also note that \(k_{tz}\) has been defined in much the same way as \(k_{iz}\) (or \(k_z\)). In other words:

\[
k_{tz} = \sqrt{k_t^2 - k_{||}^2} = \sqrt{k_t^2 - k_x^2 - k_y^2}
\]

The \(k_{||}\) appearing in the definition should not confuse you; it will soon turn out that this (together with \(k_x\) and \(k_y\)) is a conserved quantity in the processes of reflection and transmission.

We introduce two auxiliary matrices (the so-called transduction matrices):

\[
T_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Delta \theta & \sin \Delta \theta \\ 0 & -\sin \Delta \theta & \cos \Delta \theta \end{pmatrix}
\]

where \(\Delta \theta = \theta_i - \theta_t\). Notice that these two matrices are translators. The first takes the incident wave vector \(k_i\) and reflects it into the proper orientation of \(k_r\). The second rotates \(k_i\) into the direction of \(k_t\) clockwise around the \(x\) axis. Notice that neither of the tensors changes the magnitude of the wave vector. This should be done manually. (Of course, since they are in the same medium, \(k_i\) and \(k_r\) have the same magnitude.) Here is what we have just stated:

\[
\hat{k}_r = T_R \hat{k}_i \\
\hat{k}_t = T_T \hat{k}_i
\]

Here we must be extremely careful. Let’s make things clearer: the positive \(z\) axis points out from the surface. If we define it just the other way around some things will need to be
changed. We will discuss these shortly. But first convince yourself that rotating a vector clockwise (in the negative direction) is equivalent to rotating the axes counterclockwise (in the positive direction). The two dimensional rotation matrix will have the following form:

\[
R[\theta] = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}
\]

(2.8)

Rotating the vector counterclockwise is equivalent to rotating the frame clockwise, which in turn is equivalent to replacing \(\theta\) with \(-\theta\) in the above equation:

\[
R'[\theta] = R[-\theta] = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]

(2.9)

This should clarify our choice made in the definition of \(T_T\) in eq.(2.5).

To handle the magnitude problem we shall review some basic concepts:

\[
n = \frac{c}{v} = \sqrt{\frac{\epsilon \mu}{\epsilon_0 \mu_0}} = \sqrt{K_e K_m}
\]

(2.10)

where \(c\), \(\epsilon_0\) and \(\mu_0\) are the speed of light, the (electric) permittivity and the (magnetic) permeability of vacuum (or free space), respectively; \(v\), \(\epsilon\) and \(\mu\) are those of the second medium. \(n\) is called the absolute index of refraction. \(K_e\) and \(K_m\) are the relative permittivity and permeability, with obvious definitions. Notice that we can do the following manipulation:

\[
n = \frac{c}{v} = \frac{c}{\lambda \nu} = \frac{c}{2\pi \nu} \frac{2\pi \lambda}{\lambda} = \frac{ck}{\omega}
\]

In other words, the refractive index and the wave number are related through a multiplicative constant. (Recall that frequency remains unchanged even when the light beam enters a new medium.) This simply means that we can apply Snell’s law to the wave number as well as to the refractive index (keep in mind that we denote the incident, reflected and transmitted wave numbers with \(k_i\), \(k_r\), and \(k_t\) respectively; moreover \(k_i = k_r = k\)):

\[
n_i \sin \theta_i = n_t \sin \theta_t
\]

\[
k_i \sin \theta_i = k_t \sin \theta_t
\]

(2.11)

The following relations also obtain (convince yourself; keep in mind the convention regarding the sign of \(k_z\)):

\[
k_{iz} = k_i \cos \theta_i
\]

\[
k_{iy} = k_i \sin \theta_i
\]

\[
= k_{iz} \tan \theta_i
\]

\[
k_{tz} = k_t \cos \theta_t
\]

\[
k_{ty} = k_t \sin \theta_t
\]

\[
= k_{tz} \tan \theta_t
\]

\[
k_{iy} = k_{ty}
\]

(2.12)

Finally, for two non-magnetic media with permittivities \(\epsilon_1\) and \(\epsilon_2\) we have:

\[
\frac{n_2}{n_1} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{k_2}{k_1}
\]

(2.13)
Using the results above we can check for the validity of eq.(2′7). It is a relatively simple task, but is important. First notice that using eqs.(2′12) and then Snell’s law as stated in eq.(2′11) we get:

\[
\mathbf{k}_t = \begin{pmatrix}
  0 \\
  k_{ty} \\
  -k_{tz}
\end{pmatrix} = \begin{pmatrix}
  0 \\
  k_t \sin \theta_t \\
  -k_t \cos \theta_t
\end{pmatrix} = \begin{pmatrix}
  0 \\
  k_i \sin \theta_i \\
  -k_i \sin \theta_i / \tan \theta_i
\end{pmatrix}
\]

whose magnitude is simply:

\[
k_t = k_i \sin \theta_i / \sin \theta_t
\]

Thus the unit vector in the direction of the transmitted wave vector is:

\[
\hat{\mathbf{k}}_t = \frac{k_t}{k_t} = \begin{pmatrix}
  0 \\
  \sin \theta_t \\
  -\cos \theta_t
\end{pmatrix}
\]

as should have been expected. It is not difficult to see that we arrive at the same result when we carry out the following multiplication:

\[
T_T \hat{\mathbf{k}}_t = T_T \frac{k_i}{k_l} = \begin{pmatrix}
  1 & 0 & 0 \\
  \cos \Delta \theta & \sin \Delta \theta & 0 \\
  0 & -\sin \Delta \theta & \cos \Delta \theta
\end{pmatrix} \begin{pmatrix}
  0 \\
  \sin \theta_i \\
  -\cos \theta_i
\end{pmatrix}
\]

The electric field is perpendicular to the wave vector. (We shall take this rule, together with the law of reflection and Snell’s law, for granted. The interested reader should consult the literature.) That’s why we argue that we can use the same translation matrices to transform the incident electric field into the reflected and refracted fields. (We will turn to this point later again.) Of course, we shall take care of the magnitudes. For that reason we now define the new reflection and transmission matrices.

\[
R = \begin{pmatrix}
  r_{ss} & 0 & 0 \\
  0 & r_{pp} & 0 \\
  0 & 0 & r_{pp}
\end{pmatrix} \quad T = \begin{pmatrix}
  t_{ss} & 0 & 0 \\
  0 & t_{pp} & 0 \\
  0 & 0 & t_{pp}
\end{pmatrix}
\]

The elements of the matrices are the Fresnel coefficients, which are sometimes denoted by \(r_\perp\), \(t_\perp\), \(r_{||}\), \(t_{||}\) for \(r_{ss}\), \(t_{ss}\), \(r_{pp}\), \(t_{pp}\), respectively. In what follows, we will try to derive expressions for these coefficients.

With the help of these matrices we can rewrite eq.(2′2) as:

\[
(E_0 e^{ikr})_{||} + (RT_R E_0 e^{ikr})_{||} = (TT_T E_0 e^{ikr})_{||}
\]

Using the Maxwell equations we obtain the following expression for the continuity of the magnetic field:

\[
(\nabla \times E_0 e^{ikr})_{||} + (\nabla \times RT_R E_0 e^{ikr})_{||} = (\nabla \times TT_T E_0 e^{ikr})_{||}
\]

It is convenient to define two auxiliary tensors more:

\[
\nabla_x = \begin{pmatrix}
  0 & -\partial / \partial z & \partial / \partial y \\
  \partial / \partial z & 0 & -\partial / \partial x \\
  -\partial / \partial y & \partial / \partial x & 0
\end{pmatrix} \quad P_{||} = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 0
\end{pmatrix}
\]

\(2′17\)
We can now rewrite the boundary conditions as:

\[
P || E_0 e^{ikr} + P || RT R E_0 e^{ikr} = P || T T T E_0 e^{ikr} \quad (2'18)
\]

\[
P || \nabla_x E_0 e^{ikr} + P || \nabla_x RT R E_0 e^{ikr} = P || \nabla_x T T T E_0 e^{ikr} \quad (2'19)
\]

The matrices do not always commute with each other, and not all of them have an inverse, so we cannot simplify the resulting expression. The differential operator tensor acts on the exponentials accompanying the electric field, and has the following form for the three elements, from left to right:

\[
\nabla_{ix} = \begin{pmatrix} 0 & k_{iz} & k_{iy} \\ -k_{iz} & 0 & -k_{ix} \\ -k_{iy} & k_{ix} & 0 \end{pmatrix}, \quad \nabla_{rx} = \begin{pmatrix} 0 & -k_{iz} & k_{iy} \\ k_{iz} & 0 & -k_{ix} \\ -k_{iy} & k_{ix} & 0 \end{pmatrix}
\]

and \( \nabla_{tx} = \begin{pmatrix} 0 & k_{iz} & k_{ty} \\ -k_{iz} & 0 & -k_{tx} \\ -k_{ty} & k_{tx} & 0 \end{pmatrix} \)

But recalling that all \( x \) components of wave vectors vanish as a result of our assumptions, we are left with simply:

\[
\nabla_{ix} = \begin{pmatrix} 0 & k_{iz} & k_{iy} \\ -k_{iz} & 0 & 0 \\ -k_{iy} & 0 & 0 \end{pmatrix}, \quad \nabla_{rx} = \begin{pmatrix} 0 & -k_{iz} & k_{iy} \\ k_{iz} & 0 & 0 \\ -k_{iy} & 0 & 0 \end{pmatrix}
\]

and \( \nabla_{tx} = \begin{pmatrix} 0 & k_{iz} & k_{ty} \\ -k_{iz} & 0 & 0 \\ -k_{ty} & 0 & 0 \end{pmatrix} \) (2'20)

The boundary conditions must hold true at any point on the interface, so at the interface all exponentials should divide out.\(^1\) We are then left with simple matrix algebra and obtain the following four equations:

\[
0 = E_{0x}[1 + r_{ss} - t_{ss}]
\]

\[
0 = E_{0y}[1 + r_{pp} - t_{pp} \cos \Delta \theta] - E_{0z}[t_{pp} \sin \Delta \theta]
\]

\[
0 = E_{0y}[k_{iz} - r_{pp} k_{iz} - t_{pp}(k_{iz} \cos \Delta \theta - k_{ty} \sin \Delta \theta)] + E_{0z}[k_{iy} - r_{pp} k_{iy} - t_{pp}(k_{iy} \cos \Delta \theta + k_{iz} \sin \Delta \theta)]
\]

\[
0 = E_{0x}[k_{iz} - r_{ss} k_{iz} - t_{ss} k_{tz}]
\]

\(^1\)This is a tricky point and needs some attention. At the boundary the perpendicular coordinate (in this case the \( z \) coordinate) in the exponentials must vanish. This is a direct result of the requirement that the phase of the fields across the boundary should match at any point in space. (Recall that a similar requirement in time results in the conservation of frequency during the transmission process.) The remaining two coordinate dependencies in the exponentials (i.e. the \( x \) and \( y \) components) can be made to match on both sides of the boundary only through equating the two corresponding components of the wave vectors. This simply means that the parallel components of the wave vectors remain the same throughout the reflection and transmission processes. Actually, in our case the boundary is at \( z = 0 \) and the exponential term has the form \( \exp[i(k_{iz} x + k_{iy} y + k_{tz} z)], \) so the problem is resolved automatically. However, what if the exponential has the form \( \exp[i(k_{ix} x + k_{iy} y + k_{iz} z + k_{ih} h)], \) or the boundary is at \( z = h? \) Obviously the \( z \) component does not vanish at the boundary anymore. Then we must take a constant phase out (\( \exp[i k_{zh} h] \) in these cases). This part will remain unaffected by the reflection and transmission, and will appear in all electric fields besides the usual exponential part. (This situation will arise, for instance, when we consider the Weyl expansion and a boundary in the sections to follow.)
Using the first and the last equations we easily obtain:

$$r_{ss} = \frac{k_{iz}^2 - k_{tz}^2}{k_{iz}^2 + k_{tz}^2} = \frac{n_i \cos \theta_i - n_t \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t} = -\frac{\sin (\theta_i - \theta_t)}{\sin (\theta_i + \theta_t)} \quad (2'22)$$

$$t_{ss} = \frac{2k_{iz}}{k_{iz}^2 + k_{tz}^2} = \frac{2n_i \cos \theta_i}{n_i \cos \theta_i + n_t \cos \theta_t} = \frac{2 \sin \theta_i \cos \theta_i}{\sin (\theta_i + \theta_t)} \quad (2'23)$$

Solving the system of equations formed by the second and third expressions is somewhat more difficult, but still is low level algebra involving trigonometry. To avoid confusion and to aid the reader we give a simple procedure below. First recall that by requiring the $z$ axis to point up from the surface we had introduced a minus sign in the $z$ components of $\mathbf{k}_i$ and $\mathbf{k}_t$. Nevertheless, not only is this convention easier to visualize, but also it does not introduce an additional minus sign in the following equation: $E_{0z} = \tan \theta_i E_{0y}$. Convince yourself that this equation is true. (Keep in mind that the electric field is perpendicular to the wave vector.) The importance of this equation becomes obvious when we substitute it into the aforementioned system of equations. We thus obtain:

$$0 = 1 + r_{pp} - t_{pp} [\cos \Delta \theta + \sin \Delta \theta \tan \theta_i]$$

$$0 = k_{iz} (1 - r_{pp}) + k_{iy} \tan \theta_i (1 - r_{pp})$$

$$+ t_{pp} (-k_{iz} \cos \Delta \theta + k_{iy} \sin \Delta \theta) - t_{pp} \tan \theta_i (k_{ty} \cos \Delta \theta + k_{iz} \sin \Delta \theta)$$

After multiplying both sides with $\cos \theta_i$ and using some well-known trigonometric rules (the so-called addition formulas) we arrive at:

$$0 = \cos \theta_i (1 + r_{pp}) - t_{pp} \cos \theta_i$$

$$0 = (k_{iz} \cos \theta_i + k_{iy} \sin \theta_i)(1 - r_{pp})$$

$$- t_{pp} (k_{iz} \cos \theta_i + k_{iy} \sin \theta_i)$$

We can now substitute the first equation into the second one and solve:

$$0 = (k_{iz} \cos \theta_i + k_{iy} \sin \theta_i)(1 - r_{pp})$$

$$- \frac{\cos \theta_i}{\cos \theta_i} (1 + r_{pp}) (k_{iz} \cos \theta_i + k_{iy} \sin \theta_i)$$

Or:

$$r_{pp} = \frac{k_{iz} - k_{tz} + k_{iy} \tan \theta_i - k_{ty} \tan \theta_t}{k_{iz} + k_{iz} + k_{iy} \tan \theta_i + k_{ty} \tan \theta_t}$$

At this point we can follow two different routes. But we will discuss these one by one. Recalling eqs. (2'12) we can rewrite the equation in the following form:

$$r_{pp} = \frac{k_i \cos \theta_i - k_i \cos \theta_t + k_i \sin \theta_i \tan \theta_i - k_i \sin \theta_t \tan \theta_t}{k_i \cos \theta_i + k_i \cos \theta_t + k_i \sin \theta_i \tan \theta_i + k_i \sin \theta_t \tan \theta_t}$$

We use Snell’s law in the form given by eq. (2’11) and proceed:

$$r_{pp} = \frac{\cos \theta_i - \sin \theta_i \tan \theta_i + \sin \theta_i \tan \theta_i - \sin \theta_i \tan \theta_t}{\cos \theta_i + \sin \theta_i \tan \theta_i + \sin \theta_i \tan \theta_i + \sin \theta_i \tan \theta_t}$$

$$= \frac{1}{\tan \theta_i - 1} \frac{1}{\tan \theta_i} + \frac{1}{\tan \theta_i - \tan \theta_t} + \frac{1}{\tan \theta_i + \tan \theta_t}$$
\[
\frac{\tan \theta_i - \tan \theta_t}{\tan \theta_i + \tan \theta_t} (1 - \frac{\tan \theta_i \tan \theta_t}{\tan \theta_i - \tan \theta_t}) (1 - \frac{\tan \theta_i \tan \theta_t}{\tan \theta_i + \tan \theta_t}) \\
= -\frac{\tan \theta_i - \theta_t}{\tan \theta_i + \theta_t} \\
= -\tan \left( \frac{\theta_i - \theta_t}{\theta_i + \theta_t} \right) 
\]

A second approach would be to use eqs.(2'12) again, but in a slightly different manner. So:

\[
r_{pp} = \frac{k_{iz} - k_{iz} \tan^2 \theta_i - k_{iz} \tan^2 \theta_t}{k_{iz} + k_{iz} \tan^2 \theta_i + k_{iz} \tan^2 \theta_t} \\
= \frac{k_{iz} / \cos^2 \theta_i - k_{iz} / \cos^2 \theta_t}{k_{iz} / \cos^2 \theta_i + k_{iz} / \cos^2 \theta_t} \\
= \frac{k_{iz} - (\cos \theta_i / \cos \theta_t)^2 k_{iz}}{k_{iz} + (\cos \theta_i / \cos \theta_t)^2 k_{iz}} 
\]

We have the following equality, which makes use of eq.(2'13):

\[
\frac{\cos \theta_i}{\cos \theta_t} = \frac{k_{iz} k_i}{k_{iz} k_i} = \frac{k_{iz} n_t}{k_{iz} n_i} = \frac{k_{iz}}{k_{iz}} \sqrt{\epsilon_i} = \frac{k_{iz}}{k_{iz}} \sqrt{\frac{1}{\epsilon_{rel}}} 
\]

Therefore:

\[
r_{pp} = \frac{k_{iz} - (k_{iz} / k_{iz})^2 k_{iz} / \epsilon_{rel}}{k_{iz} + (k_{iz} / k_{iz})^2 k_{iz} / \epsilon_{rel}} \\
= -\frac{k_{iz} - \epsilon_{rel} k_{iz}}{k_{iz} + \epsilon_{rel} k_{iz}} 
\]

Calculation of \( t_{pp} \) is straightforward:

\[
t_{pp} = (1 + r_{pp}) \frac{\cos \theta_i}{\cos \theta_t} \\
= \left( 1 - \frac{\tan \theta_i - \theta_t}{\tan \theta_i + \theta_t} \right) \frac{\cos \theta_i}{\cos \theta_t} \\
= \left( 1 - \frac{\sin \theta_i - \theta_t \cos \theta_i + \theta_t}{\sin \theta_i + \theta_t \cos \theta_i - \theta_t} \right) \frac{\cos \theta_i}{\cos \theta_t} \\
= \left( \frac{\sin 2\theta_t}{\sin \theta_i + \theta_t \cos \theta_i - \theta_t} \right) \frac{\cos \theta_i}{\cos \theta_t} \\
= \frac{2 \sin \theta_i \cos \theta_i}{\sin \theta_i \cos \theta_i} 
\]

For the second approach we just give the result; the derivation is left as an exercise for the interested reader:

\[
t_{pp} = \frac{2 \sqrt{\epsilon_{rel}} k_{iz}}{k_{iz} + \epsilon_{rel} k_{iz}} 
\]
2.1.2 Results and Discussion

A comparison with a textbook (for instance Hecht & Zajac, Optics, or Born & Wolf, Principles of Optics) will show that our result for $r_{pp}$ has an extra minus sign at the front. This is a very good indicator that we have made our derivation properly! The minus sign arises due to our definition of $T_R$ and $R$, eqs. (2'5) and (2'14), respectively. At this point it is very appropriate to clarify several points that are likely to cause trouble.

The point is that we should leave our intuition when considering the reflection of an electromagnetic wave from a surface. We should rely only on what comes out from our mathematical derivations based on the contemporary theory, and then interpret the results and understand what is going on. It is not unusual that physical theories introduce some subtleties at some places...

First have a look at fig.(2'1) which shows our choice of the directions of the electric field components during the reflection-refraction process.

Why had we chosen such strange directions? Yes, at first look they indeed seem a little odd, but we had good reasons to think that these were the best orientations. The justification is not difficult: consider that the angle of incidence approaches zero, i.e. the incident beam strikes on the interface at an almost right angle. Then, the $z$ components of the incident and reflected waves almost vanish, and both the $y$ and the $x$ components are parallel. This could be (and indeed was) our expectation based on our intuition. But we are wrong! It just turned out that the reflection coefficients (all of them) are negative. (Note however, that $r_{pp}$ changes sign at Brewster’s angle (see fig.(2'2); so our present discussion holds true only for smaller incidence angles.) This simply means that we have to take exactly the reverse directions as these shown in fig.(2'1) for the reflected wave. Below are two figures, figs.(2'3) and (2'4) which show the actual situation.

If we want to obtain the reflection coefficients without minus signs we need to redefine $T_R$ for the electric field. Note that although this is the proper tensor which takes $k_i$ (or $k$) and transforms it into $k_r$ (or $k$), it does not transform the electric field properly. That is why a negative sign appears in $R$. So, we may define a new tensor, $T'_R$, specially for the electric field, by simply taking the minus signs from $R$ and putting them to $T_R$:

$$T'_R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2'28)$$

Yes, we may do that. But we may equally well continue with $T_R$, and bear the minus signs. In this way, we will have the same tensors for both the wave vector and the electric field. And that we do. It is a matter of taste really...

Here are the results of this section so far:

$$r_{ss} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = \frac{k_{iz} - k_{tz}}{k_{iz} + k_{tz}} \quad (2'29)$$

$$r_{pp} = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = \frac{\epsilon_{rel} k_{iz} - k_{iz}}{\epsilon_{rel} k_{iz} + k_{iz}} \quad (2'30)$$

$$t_{ss} = 1 + r_{ss} = \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_i + \theta_t)} = \frac{2k_{iz}}{k_{iz} + k_{tz}} \quad (2'31)$$

$$t_{pp} = (1 + r_{pp}) \frac{\cos \theta_i}{\cos \theta_i} = \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_i + \theta_t) \cos(\theta_i - \theta_t)} = \frac{2 \sqrt{\epsilon_{rel} k_{iz}}}{k_{iz} + \epsilon_{rel} k_{iz}} \quad (2'32)$$

Having derived the elements of the reflection and transmission matrices, $R$ and $T$ we continue with the examination of the electric fields. The total electric fields in the first and
Figure 2′1: Our convention of orientation of incident, reflected, and transmitted electric field components.

Figure 2′2: Fresnel coefficients as functions of the angle of incidence.
Fields Generated by an Oscillating Dipole

Figure 2'3: Actual orientation of incident, reflected, and transmitted electric field components. (After Brewster’s angle the $y$ and $z$ components of the reflected field reverse signs.)

Figure 2'4: Three dimensional representation of Fig. (2'3)
second media will be:
\[
E_{\text{inc}} = E_0 e^{ikr} + RT_0 E_0 e^{ikr} \\
E_{\text{tra}} = TTE_0 e^{ikr} 
\]  
(2’33)

### 2.1.3 Reflection and Transmission for Arbitrary Orientation

What if our initial assumption that the $y-z$ plane is the plane of incidence does not hold, and the $x$ components of the wave vectors are not zero anymore? Obviously we need to rotate the frame into a new orientation in which the assumptions hold, proceed with the usual calculations, and then rotate back into the original frame.

We define the angle $\varphi$ between the projection of the incident wave vector on the $x-y$ plane and the $y$ axis as:

\[
\tan \varphi = \frac{k_{ix}}{k_{iy}} 
\]  
(2’34)

Have a look at fig.(2’5) which shows the situation at hand.

![Figure 2’5: Frame alignment for arbitrary orientation.](image)

Apparently we need to rotate the frame clockwise, or equivalently, rotate the vectors counterclockwise. We do that with the help of a rotation matrix defined as:

\[
U = \begin{pmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix} 
\]  
(2’35)

The second rotation which takes us back to the original frame will be done using the inverse of the above matrix:

\[
U^{-1} = \begin{pmatrix}
\cos \varphi & \sin \varphi & 0 \\
-\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix} 
\]  
(2’36)

We can check the validity of our rotations by simply demanding that an arbitrary wave vector $k$ transform into a new wave vector $k'$ which has a zero $x$ component. Indeed, by making use of the definition of $\varphi$, eq.(2’34) we see that we have an equality of the form:

\[
\begin{pmatrix}
k'_x \\
k'_y \\
k'_z
\end{pmatrix} = \begin{pmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
k_x \\
k_y \\
k_z
\end{pmatrix}
\]
Now, if we reserve the unprimed symbols for the original frame and prime all rotated frame vectors we arrive at the following set of equations (notice that the exponentials are not affected by the rotations since they include dot products):

\[
\begin{align*}
E'_0 &= UE_0 \\
E'_{inc} &= E'_0 e^{ikr} + RT_R E'_0 e^{ikr} \\
E'_{inc} &= U^{-1}E'_{inc} \\
&= U^{-1}E'_0 e^{ikr} + U^{-1}RT_R E'_0 e^{ikr} \\
E'_t &\equiv TT_T E'_0 e^{ikr} \\
E_t &\equiv U^{-1}E'_t \\
&= U^{-1}TT_T E'_0 e^{ikr} \\
&= U^{-1}TT_T UE_0 e^{ikr} \tag{2'37}
\end{align*}
\]

Care should be taken when calculating the elements of the reflection and transmission vectors, as these still depend on the components of wave vectors which, in turn, undergo rotations. It is also important to understand that matrix \( U \) is expressed in the original frame already, so it does not belong to the same class as the other matrices.

In the above equations we can perform the following manipulation:\(^2\)

\[
\begin{align*}
E_{inc} &= E'_0 e^{ikr} + U^{-1}RT_R UE_0 e^{ikr} \\
&= E'_0 e^{ikr} + U^{-1}RUU^{-1}T_R UE_0 e^{ikr} \\
&= E'_0 e^{ikr} + RT_R E_0 e^{ikr} \\
E_t &= U^{-1}TT_T UE_0 e^{ikr} \\
&= U^{-1}TUU^{-1}T_T UE_0 e^{ikr} \\
&= T_T T_T E_0 e^{ikr} \tag{2'38}
\end{align*}
\]

where we define the following rotated versions of the matrices involved (rotated into the original frame):

\[
\begin{align*}
\mathcal{R} &= U^{-1}RU \\
\mathcal{T}_R &= U^{-1}T_R U \\
\mathcal{T} &= U^{-1}TU \\
\mathcal{T}_T &= U^{-1}T_T U \tag{2'39}
\end{align*}
\]

From this point we can proceed on to calculating these matrices. But before doing that notice that eqs.(2'29) through (2'32) giving the Fresnel coefficients (i.e. the elements of \( R \) and \( T \)), imply that a rotation around the \( z \) axis (like the rotation \( U \) that we now have) leaves these coefficients completely unchanged since they only depend on the \( z \) components of the wave vectors. We can also still use eqs.(2'12), but first we need to replace \( k_{iy} \) and \( k_{ty} \) by \( k_z \), which was defined at the very beginning right after eq.(1'75). This replacement is necessary.

\(^2\)Convince yourself that for \textit{any} three matrices \( A, B \) and \( C \) the following equation \textit{always} holds true:

\[(AB)C = A(BC)\]

This equation allows us to insert \( 1 = UU^{-1} \) in between two matrices, where obviously \( 1 \) is the identity (or unit) matrix. We will make use of this fact throughout the derivations that follow.
since the \( x \) components of the wave vectors are no longer required to be zero. Here we restate an important relation which will be of use to us in what follows:

\[
k_{mz} = \sqrt{k_m^2 - k_{||}^2}
\]

where \( m \) can take on the values \( i \) and \( t \), referring to the incident and transmitted wave vectors. Note once again that \( k_{||} \) is conserved during reflection and transmission; the same is true for \( k_{ix} \) and \( k_{iy} \), i.e. there is no need to distinguish between the \( x \) and \( y \) components of the wave vectors in the two media. This is a direct consequence of the fact that the incident, reflected and refracted wave vectors belong to the same plane (of course, together with Snell’s law).

Recall also eq.(2’13) which we rewrite here in the following form:

\[
n_m \frac{1}{\ell_0} = \frac{k_m}{\ell_0} = \sqrt{\frac{\varepsilon_m}{\ell_0}} \equiv \sqrt{\varepsilon_m} \quad (2'40)
\]

where \( \ell_0 \) is the vacuum wave number, and \( \varepsilon_m \) is redefined as the permittivity with respect to the vacuum. Thus we have:

\[
k_i = \sqrt{\varepsilon_i} k_0 \quad \text{and} \quad k_t = \sqrt{\varepsilon_t} k_0 \quad (2'41)
\]

One more word of caution about signs: keep in mind that \( k_i, k_t, k_0, \) and \( k_{||} \) are all magnitudes, i.e. are always positive; \( k_{mz} \) is positive by definition and negative \( z \) components explicitly contain a minus sign; while the \( x \) and \( y \) components of the wave vectors (in the form of \( k_{mx} \) and \( k_{my} \)) can be negative as well.

Recalling the definitions of \( U \) and \( k_{||} \) we have:

\[
U = \begin{pmatrix} k_y/k_{||} & -k_x/k_{||} & 0 \\ k_x/k_{||} & k_y/k_{||} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad U^{-1} = \begin{pmatrix} k_y/k_{||} & k_x/k_{||} & 0 \\ -k_x/k_{||} & k_y/k_{||} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2'42)
\]

Defined in this way, \( U \) takes a wave vector with arbitrary orientation and rotates it such that the \( z \) component remains the same, the \( y \) component points in the positive direction, and the \( x \) component vanishes. It is up to the reader to clarify these aspects for himself/herself.

After these preliminaries we can now proceed with the calculation of the matrices in eqs.(2’39). \( \mathcal{T}_R \) and \( \mathcal{T}_T \) (the transduction matrices) are the easiest of these equations:

\[
\mathcal{T}_R = \mathcal{T}_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (2'43)
\]

This equation will be of use in the next section...

We shall first rewrite \( \mathcal{T}_T \) in a special form which involves only \( k_{||}, k_0, \) and the permittivities of the two media.

\[
\mathcal{T}_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Delta \theta & \sin \Delta \theta \\ 0 & -\sin \Delta \theta & \cos \Delta \theta \end{pmatrix}
\]

\[
\cos \Delta \theta = \cos \theta_i \cos \theta_t + \sin \theta_i \sin \theta_t
\]
\[ e^{i\mathbf{k}_i \mathbf{x}_i} = (\mathbf{k}_{ix} \mathbf{k}_t) + (\mathbf{k}_{it} \mathbf{k}_i) \]
\[ = \left( \frac{k_{ix}}{k_i} \right) \left( \frac{k_{ix}}{k_t} \right) + \left( \frac{k_{it}}{k_i} \right) \left( \frac{k_{it}}{k_t} \right) \]
\[ = \sqrt{\frac{k_i^2 - k_{ix}^2}{k_i}} \left( \frac{k_{ix}^2 - k_{it}^2}{k_t} \right) + \left( \frac{k_{it}}{k_i} \right) \left( \frac{k_{it}}{k_t} \right) \]
\[ = \sqrt{\frac{k_i^2 - k_{ix}^2}{k_i}} \left( \frac{k_{ix}^2 - k_{it}^2}{k_t} \right) + \left( \frac{k_{it}}{k_i} \right) \left( \frac{k_{it}}{k_t} \right) \]
\[ = \gamma_i \gamma_t + \delta_i \delta_t \]

\[ \sin \Delta \theta = \sin \theta_i \cos \theta_t - \cos \theta_i \sin \theta_t \]
\[ = \left( \frac{k_{ii}}{k_i} \right) \left( \frac{k_{ix}}{k_i} \right) - \left( \frac{k_{ix}}{k_i} \right) \left( \frac{k_{it}}{k_i} \right) \]
\[ = \left( \frac{k_{ii}}{k_i} \right) \left( \sqrt{\frac{k_i^2 - k_{ix}^2}{k_i}} \right)^2 - \left( \frac{k_{ix}}{k_i} \right) \left( \frac{k_{it}}{k_i} \right) \]
\[ = \frac{k_{ii}}{k_i} \left( 1 - \frac{k_{ix}^2}{k_{it}} \right) - \frac{k_{ix}}{k_i} \left( \frac{k_{ix}^2}{k_{it}} + \frac{k_{it}^2}{k_{it}} \right) \]
\[ = \frac{k_{ii}}{k_i} \left( 1 - \frac{k_{ix}^2}{k_{it}} \right) - \frac{k_{ix}}{k_i} \left( \frac{k_{ix}^2}{k_{it}} + \frac{k_{it}^2}{k_{it}} \right) \]
\[ = \frac{k_{ii}}{\sqrt{\epsilon k_0}} \left( 1 - \frac{k_{ix}^2}{k_{it}} \right) - \frac{k_{ix}}{\sqrt{\epsilon k_0}} \left( \frac{k_{ix}^2}{k_{it}} + \frac{k_{it}^2}{k_{it}} \right) \]
\[ = \delta_i \gamma_t - \gamma_i \delta_t \quad (2'44) \]

The definitions of the variables \( \gamma \) and \( \delta \) should be obvious to the reader. Also, do not forget that \( k_0 \) is the vacuum wave number, and is related to the incident and transmitted wave numbers (i.e. the magnitudes of the incident and transmitted wave vectors) through eq. (2'41).

All that remains is to calculate \( T_T \) through:

\[ T_T = U^{-1} T_T U \]

After explicitly multiplying the matrices we obtain a tensor which has symmetric and antisymmetric parts (therefore only six independent elements):

\[ T_T = \begin{pmatrix}
\cos^2 \phi + \cos \Delta \theta \sin^2 \phi & -\cos \phi \sin \phi (1 - \cos \Delta \theta) & \sin \Delta \theta \sin \phi \\
-\cos \phi \sin \phi (1 - \cos \Delta \theta) & \sin^2 \phi + \cos \Delta \theta \cos^2 \phi & \sin \Delta \theta \cos \phi \\
-\sin \Delta \theta \sin \phi & -\sin \Delta \theta \cos \phi & \cos \Delta \theta
\end{pmatrix} \]

Note that we have used the angular definitions, so to speak, of the matrices \( U \) and \( T_T \), given by eqs. (2'35) and (2'5).

However, continuing in this way seems highly inefficient; and calculation of \( R \) and \( T \) is even more difficult. It is better to do all the rotations (i.e. the matrix multiplications) numerically, and then calculate the electric fields which will require infinite two dimensional integration...
While doing this though, it is important to keep in mind that both $R$ and $T$ are symmetric. This statement can be proved by considering an arbitrary diagonal matrix $S$, defined in the usual way:

$$S = U^{-1}SU$$

Let’s now calculate the transpose of $S$ (we make use of the fact that the transpose of the rotation matrix $U$ is also its inverse, or $U^{-1} = U^T$):

$$S^T = \left(U^T SU\right)^T = U^T S^T U = U^{-1}SU = S$$

Q.E.D.

Thus, the reflection and transmission tensors are symmetric, and we have six instead of nine independent components. This can be of use for a computer program. However, that these two tensors and the wave vector dyads themselves (appearing in the integrands of the electric fields) should all be symmetric matrices, does not necessitate that the transfer tensors be symmetric. In general, it is not true that the product of two symmetric matrices will be symmetric as well. (The reader is advised to verify this for herself/himself.)

Before we conclude this section let’s rewrite $R$ and $T$ in a slightly different form:

\[
R = \begin{pmatrix}
    r_{ss} & 0 & 0 \\
    0 & r_{pp} & 0 \\
    0 & 0 & r_{pp}
\end{pmatrix}
\]

\[
r_{ss} = \frac{k_{iz} - k_{iz}}{k_{iz} + k_{iz}} = \frac{\sqrt{k_{i}^2 - k_{z}^2} - \sqrt{k_{i}^2 - k_{z}^2}}{\sqrt{k_{i}^2 - k_{z}^2} + \sqrt{k_{i}^2 - k_{z}^2}}
\]

\[
r_{ss} = \frac{\sqrt{\epsilon_i - k_{i}^2/k_0} - \sqrt{\epsilon_t - k_{t}^2/k_0}}{\sqrt{\epsilon_i - k_{i}^2/k_0} + \sqrt{\epsilon_t - k_{t}^2/k_0}}
\]

\[
r_{pp} = \frac{\epsilon_{rel} k_{iz} - k_{iz}}{\epsilon_{rel} k_{iz} + k_{iz}} = \frac{\sqrt{\epsilon_{rel} \gamma_t - \gamma_i}}{\sqrt{\epsilon_{rel} \gamma_t + \gamma_i}}
\]

\[
T = \begin{pmatrix}
    t_{ss} & 0 & 0 \\
    0 & t_{pp} & 0 \\
    0 & 0 & t_{pp}
\end{pmatrix}
\]

And:
\[ t_{ss} = 1 + r_{ss} = \frac{2k_{iz}}{k_{iz} + k_{iz}} = \frac{2\sqrt{\varepsilon_{rel}\gamma_i}}{\sqrt{\varepsilon_{rel}\gamma_i} + \gamma_t} \]

\[ t_{pp} = (1 + r_{pp})\frac{\cos \theta_i}{\cos \theta_t} = \frac{2\sqrt{\varepsilon_{rel}k_{iz}}}{k_{iz} + \varepsilon_{rel}k_{iz}} = \frac{2\sqrt{\varepsilon_{rel}\gamma_i}}{\gamma_i + \sqrt{\varepsilon_{rel}\gamma_t}} \]  

(2'46)

The reader may like to fill in the missing steps in the second part of the above derivation. It is even possible to continue with the calculation of \( R \) and \( T \).

### 2.2 An Oscillating Dipole Above a Surface

Now we assume that the interface is the \( x - y \) plane and that a plane wave or any collection of plane waves is incident from the upper half space. We will have infinitely many plane waves (actually a two dimensional integral) for the emitting dipole case, where the dipole itself is now placed at position \( h \) on the \( z \) axis.

We can divide plane waves into two groups, according to the direction with respect to the \( z \) axis in which the waves propagate. When the \( z \) component of the wave vector is positive we say that the wave is going up, and when that is negative we say that the plane is going down. We choose to denote these fields by \( E_{up} \) and \( E_{down} \), respectively. If the component in question does not exist we will have some kind of surface waves which will not be transmitted to the second medium although they can probably penetrate into it. This is because the transmission coefficients vanish...

The up and down waves discussed above can be written in the following form using the Weyl expansion:

\[
E_{down}(r) = E_i(r) = \frac{i}{8\pi^2\varepsilon_0} \int_{-\infty}^{\infty} dk_{||} \frac{e^{i(k(r-r'))}}{k_z} \left[k^2 - k k^T\right] \mathbf{p} = \int_{-\infty}^{\infty} dk_{||} E_0(k_{||}) e^{ik(r-r')} \]

\[
E_0(k_{||}) = \frac{i}{8\pi^2\varepsilon_0} \left[k^2 - k k^T\right] \mathbf{p}/k_{iz} \quad \text{for } h > z > 0 \tag{2'47}
\]

where:

\[
r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad r' = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}
\]

i.e. \( r \) is the position in space at which we are willing to know the electric field, and \( r' \) is the position of the emitting (oscillating) dipole. Meanwhile, we remind the reader that \( k \) (or \( k_i \)) is the magnitude of the incident wave vector (i.e. is the wave number in the incidence medium). After these short remarks we continue:

\[
E_{up}(r) = \frac{i}{8\pi^2\varepsilon_0} \int_{-\infty}^{\infty} dk_{||} \frac{e^{i(k(r-r'))}}{k_z} \left[k^2 - k k^T\right] \mathbf{p} = \int_{-\infty}^{\infty} dk_{||} E_{up,0}(k_{||}) e^{ik(r-r')}
\]
\[ \mathbf{E}_{\text{up},0}(k_{||}) = \frac{i}{8\pi^2\varepsilon_0} \left[ k^2 - \mathbf{k}\mathbf{k}^T \right] \mathbf{p}/k_{iz} \]

for \( z > h \) \hspace{1cm} (2'48)

Now it is time that we define the electric fields in the two media for the dipole-surface case. Let’s denote the total field in the first medium with \( \mathbf{E}_{\text{inc}} \), and that of the second medium with \( \mathbf{E}_{\text{tra}} \). \( \mathbf{E}_{\text{tra}} \) is simply the transmitted field, and:

\[ \mathbf{E}_{\text{tra}} = \mathbf{E}_t \quad \text{for} \quad z < 0 \] \hspace{1cm} (2'49)

\( \mathbf{E}_{\text{inc}} \) on the other hand, is a summation of two other fields: the electric field directly due to the dipole which we shall denote by \( \mathbf{E}_{\text{dip}} \), and the electric field due to the surface which we denote by \( \mathbf{E}_{\text{sur}} \). \( \mathbf{E}_{\text{dip}} \) is calculated simply as if there were no surface, and \( \mathbf{E}_{\text{sur}} \) is simply the reflected field, i.e.:

\[ \mathbf{E}_{\text{sur}} = \mathbf{E}_r \] \hspace{1cm} (2'50)

In view of these definitions we have:

\[ \mathbf{E}_{\text{inc}} = \mathbf{E}_{\text{dip}} + \mathbf{E}_r \quad \text{for} \quad z > 0 \] \hspace{1cm} (2'51)

We know how to calculate \( \mathbf{E}_{\text{dip}} \); we can use any of the three expansions which all have analytic solutions. But how to calculate \( \mathbf{E}_r \) and \( \mathbf{E}_t \)? In the dipole-surface case we need to consider only the down-going waves for the reflection and transmission/refraction cases. The reflected and transmitted fields are given by eq.(2'38). To make things clearer, first consider only the integrand in eq.(2'47):

if \( \mathbf{E}_i = \mathbf{E}_0(k_{||})e^{ik_{i}r}e^{ik_{iz}h} \)
then \( \mathbf{E}_r = \mathcal{T} \mathcal{T} R \mathbf{E}_0(k_{||})e^{ik_{r}r}e^{ik_{iz}h} \)
and \( \mathbf{E}_t = \mathcal{T} \mathcal{T} \mathcal{T} \mathbf{E}_0(k_{||})e^{ik_{t}r}e^{ik_{iz}h} \) \hspace{1cm} (2'52)

Note that the phase factor due to \( r' \) has been left unaffected by the reflection and transmission. The role of this phase factor is to move the dipole in space, and has no relevance to what is going on at the surface. Also remember that \( k_{iz} < 0 \) and \( k_{tz} < 0 \); see, for instance, eq.(2'4).

Since in reality we have an infinite collection of waves we need to place integrals to obtain:

if \( \mathbf{E}_i = \int_{-\infty}^{\infty} dk_{||} \mathbf{E}_0(k_{||})e^{ik_{r}r}e^{ik_{iz}h} \)
then \( \mathbf{E}_r = \int_{-\infty}^{\infty} dk_{||} \mathcal{T} \mathcal{T} \mathcal{T} \mathbf{E}_0(k_{||})e^{ik_{r}r}e^{ik_{iz}h} \)
and \( \mathbf{E}_t = \int_{-\infty}^{\infty} dk_{||} \mathcal{T} \mathcal{T} \mathbf{E}_0(k_{||})e^{ik_{t}r}e^{ik_{iz}h} \) \hspace{1cm} (2'53)

2.2.1 The Reflected Electric Field

We are now at a position to investigate the transfer tensor. Our first concern will be the upper half space. We make the following definitions analogous to those made previously for the electric field.

\[ \mathbf{E}_{\text{inc}} = \mathbf{E}_{\text{dip}} + \mathbf{E}_r \]
\[ T_{\text{inc}} \mathbf{p} = T_{\text{dip}} \mathbf{p} + T_{\text{r}} \mathbf{p} \] \hspace{1cm} (2'54)
$T_{dip}$ is what we can calculate analytically. The reflected transfer tensor is somewhat more tedious. In view of eqs. (2'47) and (2'53) we have:

$$T_r = \frac{i}{8\pi^2 \epsilon_0} \int_{-\infty}^{\infty} dk ||R T_R [k^2 - kk^T] e^{ikr} e^{ikz h} / kiz$$ (2'55)

It is possible to put this equation in a radically different form, which might be of use. Since $T_R$ is the same as $T_{R'}$, the following equation holds. (The reader should verify the validity of this important relation herself/himself.)

$$T_R [k^2 - kk^T] = [k^2 - kk^T] T_R$$ (2'56)

Using this equation we obtain:

$$\mathbf{E}_r = T_r \mathbf{p} = \left( \frac{i}{8\pi^2 \epsilon_0} \int_{-\infty}^{\infty} dk ||R [k^2 - kk^T] e^{ikr + ikz h} / kiz \right) T_R \mathbf{p}$$

(2'57)

where $\mathbf{p}$ is some kind of reflected dipole moment. We can also write:

$$\mathbf{E}_r = \left( \frac{i}{8\pi^2 \epsilon_0} \int_{-\infty}^{\infty} dk ||R [k^2 - kk^T] e^{ik(r-r')} / kiz \right) \mathbf{p}$$

(2'58)

where:

$$r' = \begin{pmatrix} 0 \\ 0 \\ -h \end{pmatrix}$$

(2'59)

and is also reflected.

If the dipole is not on the z axis then replace $r'$ and $r''$ by:

$$r' = \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} \quad \text{and} \quad r'' = \begin{pmatrix} h_x \\ h_y \\ -h_z \end{pmatrix}$$

(2'60)

respectively. In any case the following equations hold:

$$r' = T_R r'' \quad \text{and} \quad \mathbf{p} = T_R \mathbf{p}$$

(2'61)

The interpretation of these results follows. In the dipole-surface case, for a point in the upper half space (i.e. a point above the surface), the electric field is given simply as the sum of two separate fields: the field directly due to the oscillating dipole which is at position $h$ on the z axis, and the field due to an imaginary reflected dipole located at $z = -h$ below the surface. Viewed in this way, $\mathbf{E}_r$ can be said to have the expected classical aspects of a reflected field...

### 2.2.2 The Transmitted Electric Field

For the lower half space we have simply:

$$\mathbf{E}_{tr} = \mathbf{E}_t = T_t \mathbf{p}$$

(2'62)
We can write the transmitted transfer tensor using eqs. (2’47) and (2’53) as:

\[
T_t = \frac{i}{8\pi^2\epsilon_0} \int_{-\infty}^{\infty} d\mathbf{k}_t \mathbf{T}^T T \left[ k^2 - \mathbf{k}\mathbf{k}^T \right] e^{i\mathbf{k}_t r} e^{ik_i h/k_{iz}} \tag{2’63}
\]

But we would like to have a picture similar to the reflected electric field, i.e. we would like to part from reality and play with imaginary results. The passage from the former to the latter is not obvious. Let’s first define a new matrix:

\[
\mathbf{K}_i = \mathbf{k}_i \mathbf{k}_i^T \tag{2’64}
\]

We have explicitly indicated that the wave numbers involved are in the upper medium. Suppose that we have:

\[
\mathbf{K}_i = U^{-1} \mathbf{K} U \tag{2’65}
\]

or, in other words:

\[
\mathbf{K}_i = U \mathbf{K}_i U^{-1} \tag{2’66}
\]

Then:

\[
\mathbf{T}_T \mathbf{K}_i = U^{-1} \mathbf{T}_T \mathbf{U} U^{-1} \mathbf{K} \mathbf{U}
\]

\[
= U^{-1} \mathbf{T}_T \mathbf{K}_i \mathbf{U} \tag{2’67}
\]

Let’s now calculate the matrix \( \mathbf{K}_i \):

\[
\mathbf{K}_i = U \mathbf{K}_i U^{-1}
\]

\[
= \begin{pmatrix}
  k_y/k_{||} & -k_x/k_{||} & 0 \\
  k_x/k_{||} & k_y/k_{||} & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  k_x^2 & k_x k_y & -k_x k_{iz} \\
  k_x k_y & k_y^2 & -k_y k_{iz} \\
  -k_{iz} k_x & -k_{iz} k_y & k_{iz}^2
\end{pmatrix}
\begin{pmatrix}
  k_y/k_{||} & k_x/k_{||} & 0 \\
  -k_x/k_{||} & k_y/k_{||} & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  0 & 0 & 0 \\
  0 & k_{||}^2 & -k_{iz} k_{||} \\
  0 & -k_{iz} k_{||} & k_{iz}^2
\end{pmatrix} \tag{2’68}
\]

Note that all \( x \) components have effectively vanished.

We would like to express matrix \( \mathbf{T}_T \) in a similar form, so we use intermediate results from eq. (2’44):

\[
\mathbf{T}_T = \frac{1}{k_i k_t} \begin{pmatrix}
  k_i k_t & 0 & 0 \\
  0 & k_i k_t + k_{iz}^2 & k_{||}(k_{iz} - k_{iz}) \\
  0 & -k_{||}(k_{iz} - k_{iz}) & k_{iz} k_{iz} + k_{||}^2
\end{pmatrix} \tag{2’69}
\]

Now consider the following product:

\[
\mathbf{T}_T \left[ k_i^2 - \mathbf{K}_i \right] = \frac{1}{k_i k_t} \begin{pmatrix}
  k_i k_t & 0 & 0 \\
  0 & k_i k_t + k_{iz}^2 & k_{||}(k_{iz} - k_{iz}) \\
  0 & -k_{||}(k_{iz} - k_{iz}) & k_{iz} k_{iz} + k_{||}^2
\end{pmatrix}
\begin{pmatrix}
  k_i^2 & 0 & 0 \\
  0 & k_{iz}^2 & k_{iz} k_{||} \\
  0 & 0 & k_{||}^2
\end{pmatrix} \tag{2’70}
\]
Obviously we think of \( k_i^2 \) as \( k_i^2 I \), where 1 is the 3 \times 3 unit (or identity) matrix. Why we have considered this particular product will become clear soon. But first we want to examine another product of the same form:

\[
\left[ k_i^2 - K_i \right] T_T = \frac{1}{k_i k_t} \left( \begin{array}{ccc}
 k_i^2 & 0 & 0 \\
 0 & k_{iz} & k_{iz} \\
 0 & k_{iz} & k_i^2 \\
\end{array} \right) \left( \begin{array}{ccc}
 k_i k_t & 0 & 0 \\
 0 & k_{iz} k_t + k_{iz}^2 & k_i (k_{iz} - k_{iz}) \\
 0 & -k_{iz} (k_{iz} - k_{iz}) & k_{iz} k_t + k_{iz}^2 \\
\end{array} \right)
\]

\[
= k_t k_i \left( \begin{array}{ccc}
 k_i k_t & 0 & 0 \\
 0 & k_{iz} k_t & k_{iz}^2 \\
 0 & k_{iz} k_t & k_i k_t \\
\end{array} \right)
\]

We thus arrive at the following equation:

\[
k_i T_T \left[ k_i^2 - K_i \right] = k_t \left[ k_i^2 - K_i \right] T_T \quad \text{or:}
\]

\[
T_T \left[ k_i^2 - K_i \right] = \left( k_t \right)^2 \left[ k_i^2 - K_i \right] T_T
\]

\[
= \epsilon_{rel} \left[ k_i^2 - K_i \right] T_T
\]

We shall make use of this equation in the following form, which is similar to what we have previously done in eq.(2'67):

\[
T_T \left[ k_i^2 - K_i \right] = U^{-1} T_T U U^{-1} \left[ k_i^2 1 - K_i \right] U
\]

\[
= U^{-1} T_T \left[ k_i^2 1 - K_i \right] U
\]

\[
= U^{-1} \epsilon_{rel} \left[ k_i^2 1 - K_t \right] T_T U
\]

\[
= \epsilon_{rel} U^{-1} \left[ k_i^2 1 - K_t \right] U U^{-1} T_T U
\]

\[
= \epsilon_{rel} \left[ k_i^2 - K_t \right] T_T
\]

\[\text{(2'73)}\]

\( K_t \) is defined in much the same way as \( K_i \):

\[
K_t = k_t k_i^T = \left( \begin{array}{ccc}
 k_i^2 & k_x k_y & -k_x k_{iz} \\
 k_y k_x & k_i^2 & -k_y k_{iz} \\
 -k_{iz} k_x & -k_{iz} k_y & k_{iz}^2 \\
\end{array} \right)
\]

\[\text{(2'74)}\]

We can now return to eq.(2'62):

\[
E_t = T_t p = \frac{i}{8 \pi^2 \epsilon_0} \int_{-\infty}^{\infty} d k_{iz} |T| T_T \left[ k_i^2 - K_i \right] \frac{e^{ik_r r + ik_{iz} h}}{k_{iz}} p
\]

\[
= \frac{i}{8 \pi^2 \epsilon_0} \int_{-\infty}^{\infty} d k_{iz} |T| \left[ k_i^2 - K_i \right] \frac{e^{ik_r r + ik_{iz} h}}{k_{iz}} (\epsilon_{rel} T T) p
\]

\[
= \frac{i}{8 \pi^2 \epsilon_0} \int_{-\infty}^{\infty} d k_{iz} |T| \left[ k_i^2 - K_i \right] \frac{e^{ik_r r + ik_{iz} h}}{k_{iz}} p_t
\]

\[\text{(2'75)}\]

where \( p_t \) is some kind of transmitted dipole moment. However, note that \( p_t \) now depends on the variables over which we perform an infinite two dimensional integration. In other words, what we did was not much of a simplification. It is rather questionable whether we should stick to this form, or the previous form given through eqs.(2'62) and (2'63).
2.2.3 Results

Perhaps it is a good idea to summarize our results of this section. First for the medium of incidence:

\[ E_{inc} = E_{dp} + E_r \]

\[ E_r = \left( -\frac{i}{8\pi^2\varepsilon_0} \int_{-\infty}^{\infty} dk_\parallel \mathcal{R} \mathcal{T} \left[ k_i^2 - k_\parallel k_T^T \right] \frac{e^{ikr+i\varepsilon h}}{k_{iz}} \right) \mathbf{p} \text{ or:} \]

\[ E_r = \left( -\frac{i}{8\pi^2\varepsilon_0} \int_{-\infty}^{\infty} dk_\parallel \mathcal{R} \left[ k_i^2 - k_\parallel k_T^T \right] \frac{e^{ikr+i\varepsilon h}}{k_{iz}} \right) \mathbf{p} \quad (2'76) \]

Then for the medium of transmission:

\[ E_{tra} = E_t \]

\[ E_t = \left( -\frac{i}{8\pi^2\varepsilon_0} \int_{-\infty}^{\infty} dk_\parallel \mathcal{T} \mathcal{T} \left[ k_i^2 - k_\parallel k_T^T \right] \frac{e^{ikr+i\varepsilon h}}{k_{iz}} \right) \mathbf{p} \text{ or:} \]

\[ E_t = \left( -\frac{i}{8\pi^2\varepsilon_0} \int_{-\infty}^{\infty} dk_\parallel \mathcal{T} \left[ k_i^2 - k_\parallel k_T^T \right] \frac{e^{ikr+i\varepsilon h}}{k_{iz}} p_t \right) \quad (2'77) \]

2.2.4 The Dipole-Surface Case in Cylindrical Coordinates

In this section we will rewrite the equations that we have found in cylindrical coordinates (i.e. in terms of \( k_\parallel \), \( \theta \), and \( k_{iz} \)). Our intention is to simplify the integrations and facilitate numerical implementation. First note that \( U \) as given in eq.(2'35) is already in some kind of polar coordinates; all we need do is replace \( \varphi \) by \( \pi/2 - \theta \):

\[ U = \begin{pmatrix} \sin \theta & -\cos \theta & 0 \\ \cos \theta & \sin \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2'78) \]

The definition of the inverse should be obvious.

The reflection matrix \( R \) is left unchanged in the cylindrical coordinates, since its elements (i.e. \( r_{ss} \) and \( r_{pp} \)) depend only on \( z \) components. So, we can still use eq.(2'45) as a definition. The same is true for \( T \), eq.(2'46)...

The rest comes naturally (we use eq.(2'58) for the reflected electric field):

\[ E_r = \left( \frac{i}{8\pi^2\varepsilon_0} \int_0^{2\pi} \int_0^\infty k_\parallel dk_\parallel d\theta \mathcal{R} \left[ k_i^2 - k_\parallel k_T^T \right] \frac{\exp \left[ if(k_\parallel, \theta) \right]}{k_{iz}(k_\parallel)} \right) \mathbf{p} \quad (2'79) \]

where:

\[ f(k_\parallel, \theta) = k(\mathbf{r} - \mathbf{r}') = x k_\parallel \cos \theta + y k_\parallel \sin \theta + (z + h) k_{iz}(k_\parallel) \]

\[ = x k_\parallel \cos \theta + y k_\parallel \sin \theta + (z + h) \sqrt{k_i^2 - k_\parallel^2} \]

\[ \left[ k_i^2 - k_\parallel k_T^T \right] = \begin{pmatrix} k_i^2 - k_\parallel^2 \cos^2 \theta & -k_i^2 \cos \theta \sin \theta & -k_\parallel k_{iz} \cos \theta \\ -k_i^2 \cos \theta \sin \theta & k_i^2 - k_\parallel^2 \sin^2 \theta & -k_\parallel k_{iz} \sin \theta \\ -k_\parallel k_{iz} \cos \theta & -k_\parallel k_{iz} \sin \theta & k_{iz}^2 \end{pmatrix} \quad (2'80) \]

For the transmitted field we shall use eq.(2'63) instead of eq.(2'75):

\[ E_t = \left( \frac{i}{8\pi^2\varepsilon_0} \int_0^{2\pi} \int_0^\infty k_\parallel dk_\parallel d\theta \mathcal{T} \mathcal{T} \left[ k_i^2 - k_\parallel k_T^T \right] \frac{\exp \left[ ig(k_\parallel, \theta) \right]}{k_{iz}(k_\parallel)} \right) \mathbf{p} \quad (2'81) \]
where:

\[ g(k_{||}, \theta) = k_{\|} x \cos \theta + y k_{\|} \sin \theta - z k_{||} \delta_{i} (k_{||}) + h k_{\|} (k_{||}) \]

\[ k_{i z}(k_{||}) = \sqrt{k_{i}^{2} - k_{||}^{2}} = k_{i} \sqrt{1 - k_{||}^{2}/k_{i}^{2}} = k_{i} \gamma_{i} \]

\[ k_{t z}(k_{||}) = \sqrt{k_{t}^{2} - k_{||}^{2}} = k_{t} \sqrt{1 - k_{||}^{2}/k_{t}^{2}} = k_{t} \gamma_{t} \]

\[ \left[ k_{i}^{2} - k_{\|} k_{T}^{T} \right] = \begin{pmatrix} k_{i}^{2} \cos^{2} \theta & -k_{i}^{2} \cos \theta \sin \theta & k_{i} \gamma_{i} \cos \theta \\ -k_{i}^{2} \cos \theta \sin \theta & k_{i}^{2} - k_{i}^{2} \sin^{2} \theta & k_{i} \gamma_{i} \sin \theta \\ k_{i} \gamma_{i} \cos \theta & k_{i} \gamma_{i} \sin \theta & k_{i}^{2} \end{pmatrix} \]

(2’82)

Note that at \( k_{||} = k_{i} \) we have a singularity, which is now described in a far better way than it used to be with the cartesian coordinates. Also note that at this point \( k_{iz} \) which itself is a function of \( k_{||} \) becomes imaginary. This, in turn, means that a part of the exponential becomes negative. Thus the integrand decreases exponentially and the integral converges. The cylindrical form seems beneficial; all that remains is to prove this statement by making good use of the result...

The wave vector dyads as given in eqs.(2’80) and (2’82) are quite awesome. It could be preferable to have them in a form similar to \( K_{i} \) in eqs.(2’66) and (2’68). We can do that by making the following replacements (the definitions of \( \mathcal{M}_{i} \) and \( \mathcal{M}_{i} \) should be obvious from what follows):

\[ \mathcal{R} \left[ k_{i}^{2} - k_{\|} k_{T}^{T} \right] = \mathcal{R} \mathcal{M}_{i} = U^{-1} \mathcal{R} \mathcal{M}_{i} U \]

\[ \mathcal{T} \mathcal{T}^{T} \left[ k_{i}^{2} - k_{\|} k_{T}^{T} \right] = \mathcal{T} \mathcal{T}^{T} \mathcal{M}_{i} = U^{-1} \mathcal{T} \mathcal{T}^{T} \mathcal{M}_{i} U \]

(2’83)

where:

\[ \mathcal{M}_{i} = \begin{pmatrix} k_{i}^{2} & 0 & 0 \\ 0 & k_{iz}^{2} & -k_{iz} k_{||} \\ 0 & -k_{iz} k_{||} & k_{||}^{2} \end{pmatrix} \]

and

\[ \mathcal{M}_{i} = \begin{pmatrix} k_{i}^{2} & 0 & 0 \\ 0 & k_{iz}^{2} & k_{iz} k_{||} \\ 0 & k_{iz} k_{||} & k_{||}^{2} \end{pmatrix} \]

(2’84)

We believe this is a much better formulation...
Chapter 3

The Dipole-Surface Case II: Cylindrical Waves

3.1 Treatment of the Dipole-Surface Problem with Cylindrical Waves

We discussed the effects of placing an oscillating dipole near a surface in the previous chapter using plane waves. In this chapter we will try to do the same, but this time using cylindrical waves. As was discussed in the beginning of this text, although we call them cylindrical, these waves are nothing but a special kind of plane waves. So, theoretically we can apply the same rules.

The electric field is now given through eq.(1′31) which we rewrite here:

\[
E(\mathbf{r}) = \frac{i}{4\pi\epsilon_0} \int_0^\infty dk_{||} \frac{k_{||}}{k_{iz}} \left[ \nabla \nabla^T + k_i^2 \right] J_0(k_{||}\rho) e^{ik_{||}|z-h|} \mathbf{p}_j
\] (3′1)

After explicitly carrying out the differentiations we finally arrive at eq.(1′34):

\[
E(\mathbf{r}) = \left[ T_1 \hat{\mathbf{1}} + T_{pp} \hat{\mathbf{p}}^T + T_{zp} \hat{\mathbf{z}}^T + T_{pz} \hat{\mathbf{p}} \hat{\mathbf{z}}^T + T_{||} \hat{\mathbf{1}}_{||} \right] \mathbf{p}_j
\] (3′2)

The transfer tensor as given in this equation is a summation of four terms, each consisting of a 3 × 3 tensor and a coefficient to be integrated over one variable (\(k_{||}\)). The integrations can be rewritten in slightly simpler forms given through eqs.(1′51-1′56). Actually, since only the z component of the wave vector changes when the incident field is transmitted into the second medium and all the integrations are one dimensional, our task is relatively easy as compared to the somewhat awkward Weyl (plane wave) expansion. Further, we can treat each of the constituents of the transfer tensor separately and then sum up to obtain the total incident, reflected and transmitted fields...

We begin with considering a single incident wave of the form:

\[
E_i(\mathbf{r}) = E_0(k_{||}) e^{ik_{||}|z-h|}
\] (3′3)

The amplitude coefficient \(E_0(k_{||})\) will have different forms for different terms making up the transfer tensor. Just like in the plane wave expansion we will consider only down going
waves as incident (in which case we have \( z < h \)). Then, much similar to what we have done in eqs.(2’52) and (2’53), we have:

If \( E_i = E_0(k_{||}) e^{-ik_{iz}z} e^{ik_{iz}h} \)

then \( E_r = RT_R E_0(k_{||}) e^{ik_{iz}z} e^{ik_{iz}h} \)

and \( E_t = TT_T E_0(k_{||}) e^{-ik_{iz}z} e^{ik_{iz}h} \) \( (3’4) \)

Note that we have used the unrotated versions of the reflection and transmission tensors. Our argument is simple: since we are now dealing with plane waves parallel to the \( x - y \) plane we do not have to bother about rotating frames back and forth.

Further simplifications are also possible. Recall the definitions of the reflection, transmission and transduction matrices given through eqs.(2’5) and (2’14). For normal incidence the transmission transduction matrix \( T_T \) becomes just the unit matrix. The Fresnel coefficients as given by eqs.(2’22), (2’23), (2’24) and (2’26), on the other hand, become:

\[
\begin{align*}
r_{pp} &= -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = r_{ss} \\
&= \frac{k_{iz} - k_{tz}}{k_{iz} + k_{tz}} = r \\
t_{pp} &= t_{ss} = 1 + r = \frac{2k_{iz}}{k_{iz} + k_{tz}} = t \quad \text{(3’5)}
\end{align*}
\]

It is important to note that both \( k_{iz} \) and \( k_{tz} \) are functions of \( k_{||} \). This means that effectively the properties of the two media are different for each separate wave appearing in the integrand. (Only in this way can we explain the functional dependence of the reflection and transmission coefficients.) It is convenient to define new simpler transmission and reflection matrices:

\[
R = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & -r \end{pmatrix} = r(k_{||})I', \quad T = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} = t(k_{||})I \quad \text{(3’6)}
\]

The definitions of the matrices \( I \) and \( I' \) should be more than obvious.

We can now continue with our real task of calculating the electric fields in the two media. At this point we make a radical change in notation in order to prevent any kind of confusion. We write the incident, reflected and transmitted fields in the following way:

\[
\begin{align*}
E_i &= I p_j = \left( I_1 + I_{pp} + I_{zp} + I_{||} \right) p_j \\
E_r &= R p_j = \left( R_1 + R_{pp} + R_{zp} + R_{||} \right) p_j \\
E_t &= T p_j = \left( T_1 + T_{pp} + T_{zp} + T_{||} \right) p_j \quad \text{(3’7)}
\end{align*}
\]

where:

\[
\begin{align*}
I_1 &= T_1 \\
I_{pp} &= T_{pp} \hat{\rho}^T p_j \\
I_{zp} &= T_{zp} (\hat{z} \hat{\rho}^T + \hat{\rho} \hat{z}^T) \\
I_{||} &= T_{||} \hat{1} \quad \text{(3’8)}
\end{align*}
\]

and so on.
To calculate all the coefficients for the incident field we need only three integrations. This was the final result that we could obtain in the simplification of the integrals, without losing the inherent shape of the waves. However, now we will change the integrands and thus will obtain an integral of a $3 \times 3$ tensor (the transfer tensor). The tensor is not symmetric, so we are left with nine integrations. This is unavoidable at the moment. But we shall still stick to the refined results since they do not have the singularities caused by a $k_z$ in the denominator and do not bring about any extra work. The $z = h$ case is problematic even for the unrefined integrations, eqs.(1'51-1'56). But since we suppose that the reader is a down-to-earth person, we think he/she will calculate the free dipole field analytically (which has no problems in itself unless one tries to figure out the electric field at the position of the dipole). It will turn out that we will not have a singularity for the reflection case (expectedly, the $|z - h|$ in the denominator of the equations should mutate into a $|z + h|$), and for the transmission case $z = h$ is a very difficult configuration to visualize (unless we violate a few physical laws).

It now pays off. The extra amount of work that we did in the cylindrical wave expansion for the free dipole now illuminates the way before us...

We will now study the $I_1$ terms in eq.(3'7) in some detail. For this term of the free dipole transfer tensor we had found the following results:

$$T_1 = \frac{2}{|z - h|} \chi_0 - \frac{\rho}{|z - h|} \chi_1$$

where:

$$\chi_\nu = \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\parallel |\Psi_\nu|^2$$

$$\Psi_\nu = k_\parallel^{\nu+1} J_\nu(k_\parallel \rho) e^{ik_\parallel z}|z - h|$$

Now, in view of eq.(3'4) we obtain:

$$I_1 = \frac{1}{4\pi\epsilon_0} \frac{1}{h - z} \int_0^\infty dk_\parallel \left( 2k_\parallel J_0(k_\parallel \rho) - k_\parallel^2 \rho J_1(k_\parallel \rho) \right) T e^{-ik_\parallel z} e^{ik_\parallel z h}$$

$$R_1 = \frac{1}{4\pi\epsilon_0} \frac{1}{h - z} \int_0^\infty dk_\parallel \left( 2k_\parallel J_0(k_\parallel \rho) - k_\parallel^2 \rho J_1(k_\parallel \rho) \right) R e^{ik_\parallel z} e^{ik_\parallel z h}$$

$$T_1 = \frac{1}{4\pi\epsilon_0} \frac{1}{h - z} \int_0^\infty dk_\parallel \left( 2k_\parallel J_0(k_\parallel \rho) - k_\parallel^2 \rho J_1(k_\parallel \rho) \right) T e^{-ik_\parallel z} e^{ik_\parallel z h}$$

The first of these is just restatement of eq.(1'48); only keep in mind that for a wave to be incident we need $z$ to be less than $h$. Note that we have replaced $I_1$ by $I$. That is for visual purposes only...

Is what we have just done true? Acceptable answers are “maybe yes, maybe no” and “so, so”. But to be precise we should say “not at all”!

### 3.1.1 Why It Does not Work

We have three formulation layers of the electric field (to be discussed shortly). The problem we are now facing is at what level reflection and transmission need be introduced.

Recall that the electric field of a dipole is given through a differentiation of a certain expansion. That this indeed is true can be best seen from eqs.(1'19), (1'31) and (1'76). This we shall refer to as the first layer of formulation. Introducing reflection and transmission
at this level would cause severe oddities in the resulting formulas. Keep in mind that the \( z \) component of the wave vector changes (during reflection), and that this component appears multiplied by a spatial coordinate in the exponentials. This simply means that the differentiations will bring these modifications in the phase down to the \textit{magnitude} part of the incident electric field. In other words, not only will we apply our reflection and transmission tensors together with the transduction matrices but also we will effectively introduce some \textit{undesirable} extra terms. The first layer of formulation is just too early...\(^1\)

By \textit{refining} the integrations obtained as a result of applying explicitly the gradients in the Sommerfeld expansion, i.e. by converting eqs.(1’34-1’38) to eqs.(1’51-1’56), we have introduced a new layer of differentiation into our formulas. Written in this new form, the formulas will be referred to as the \textit{third layer of formulation}. (Obviously, the intermediate form, or the direct results obtained from the calculation of the gradients is the missing \textit{second layer of formulation}.) Much similar to the first, the third formulation causes problems related to differentiation. Interestingly (and fortunately), the reflected field can still be \textit{refined} through the same steps followed before. Apart from a change of sign of \( z \), this will not cause any other problem.

However, for the transmitted field the situation is not so trivial. Not the sign, but the coefficient of \( z \) changes (from \( k_{iz} \) to \( k_{iz} \)). Simplifications in the integration and especially removal of the singularity are no longer feasible. We have to use eqs.(1’34-1’38) to calculate the transmitted field...

\section{3.1.2 How It Should Work}

Above we tried to give a rough feeling so as to why we have to make use of both eqs.(1’34-1’38) and eqs.(1’51-1’56). Now we can turn back to the derivations...

In view of what has been done and what has been said, the rest comes naturally (and is left as an exercise to the reader\(^2\)):

\[
\begin{align*}
\textbf{R}_1 & = \frac{1}{4\pi\varepsilon_0} \int_0^\infty dk_z \left[ \frac{1}{h + z} \left( 2k_z J_0(k_z \rho) - k_z^2 \rho J_1(k_z \rho) \right) \right] \text{RI} e^{ik_z z} e^{ik_z h} \\
\textbf{R}_{pp} & = \frac{1}{4\pi\varepsilon_0} \int_0^\infty dk_z \left[ \frac{\rho}{h + z} k_z^2 J_1(k_z \rho) \right] \hat{\rho} \hat{\rho}^T e^{ik_z z} e^{ik_z h} \\
\textbf{R}_{zp} & = \frac{1}{4\pi\varepsilon_0} \int_0^\infty dk_z \left[ -k_z^2 J_1(k_z \rho) \right] \text{RI} (\hat{z} \hat{\rho}^T + \hat{\rho} \hat{z}^T) e^{ik_z z} e^{ik_z h} \\
\textbf{R}_\parallel & = \frac{1}{4\pi\varepsilon_0} \int_0^\infty dk_z \left[ (ik_z - \frac{1}{h + z}) k_z J_0(k_z \rho) \right] \text{RI} e^{ik_z z} e^{ik_z h} \\
\textbf{T}_1 & = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk_z \left[ \frac{k_z^3}{k_z J_0(k_z \rho)} \right] \text{TO} e^{-ik_z z} e^{ik_z h} \\
\textbf{T}_{pp} & = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk_z \left[ \frac{k_z^3}{k_z J_0(k_z \rho)} \right] \hat{\rho} \hat{\rho}^T e^{-ik_z z} e^{ik_z h} \\
\textbf{T}_{zp} & = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk_z \left[ ik_z^2 J_1(k_z \rho) \right] \text{TO} (\hat{z} \hat{\rho}^T + \hat{\rho} \hat{z}^T) e^{-ik_z z} e^{ik_z h} \\
\textbf{T}_\parallel & = \frac{i}{4\pi\varepsilon_0} \int_0^\infty dk_z \left[ k_z J_0(k_z \rho) - \frac{k_z^2}{k_z \rho} J_1(k_z \rho) \right] \text{TI} e^{-ik_z z} e^{ik_z h}
\end{align*}
\]

\(^1\)If you have not got the point here (i.e. if the server is down or not responding), feel free to re-read the passage once again.

\(^2\)Note that we try to give the reader as much exercise as possible, to keep him/her fit and alert.
To find the total reflected and transmitted fields we just need to sum the individual components, as in eq.(3’7). To avoid division by zero for the \( \rho = 0 \) case we may apply the same trick as was done previously with the help of eq.(1’38)...

3.2 Boundary Conditions for the Vector Potential

In this section we will try to derive reflection and transmission coefficients for the vector potential. But before doing that it might be better to repeat the derivations of Novotny[^1], who makes use of the Hertz potential. However, since the vector potential and the Hertz potential are connected through a constant (for time-harmonic fields) we can always go back to the vector potential.

3.2.1 The Hertz Potential

At the boundary the tangential components of the electric and magnetic fields should be continuous. We know the relations between the Hertz potential and the fields from eq.(1’24). We will stick to a vertical and then a horizontal dipole, therefore we shall adopt Novotny’s idea that only two components for the Hertz potential are enough to describe the fields[^2]:

\[
\Pi = \begin{pmatrix}
\Pi_x \\
0 \\
\Pi_z
\end{pmatrix}
\]  

(3’11)

We will denote functions belonging to the first (incidence) medium by the subscript 1, and those belonging to the second medium with the subscript 2. Then the boundary conditions become:

\[
E_{1t} = E_{2t} \\
H_{1t} = H_{2t}
\]  

(3’12)

These become upon substituting eq.(1’24):

\[
[(k_1^2 + \nabla \nabla \cdot) \Pi_1]_t = [(k_2^2 + \nabla \nabla \cdot) \Pi_2]_t \\
\varepsilon_1 [\nabla \times \Pi_1]_t = \varepsilon_2 [\nabla \times \Pi_2]_t
\]  

(3’13)

Rewriting these equations for the components of the Hertz potential we arrive at the four boundary conditions for the total of four unknown components:

\[
k_1^2 \Pi_{1x} + \frac{\partial}{\partial x} \nabla \cdot \Pi_1 = \frac{\partial}{\partial x} \nabla \cdot \Pi_2 + k_2^2 \Pi_{2x} \\
\frac{\partial}{\partial y} \nabla \cdot \Pi_1 = \frac{\partial}{\partial y} \nabla \cdot \Pi_2 \\
\varepsilon_1 \frac{\partial \Pi_{1z}}{\partial y} = \varepsilon_2 \frac{\partial \Pi_{2z}}{\partial y} \\
\varepsilon_1 \frac{\partial \Pi_{1x}}{\partial z} = \varepsilon_2 \frac{\partial \Pi_{2x}}{\partial z}
\]  

(3’14)

[^1]: One can argue that since we confine the dipole to a plane which is two dimensional, only two components are required for the potential. We are actually making use of the symmetry of the problem. In particular, when we consider a vertical dipole, the field will be symmetric around the \( z \) axis. The only change in the electric field will be along the \( z \) axis; and therefore there will be a potential change only in this direction. This could mean that we need only one component (the \( z \) component) for the Hertz potential in the vertical dipole case.
Meanwhile, note that if we assume the permeabilities of the two media to be the same, then:

\[ \varepsilon_n = \frac{1}{\mu_1 n^2} = \frac{k_n^2}{\mu \omega^2} \]  

(3'15)

i.e. the permittivity and the wave number are related through a constant.

The other two boundary conditions\(^4\) yield:

\[ \varepsilon_1 \left[ (k_1^2 + \nabla \nabla \cdot \Pi_1) \right]_n = \varepsilon_2 \left[ (k_2^2 + \nabla \nabla \cdot \Pi_2) \right]_n \]

\[ \varepsilon_1 \left( k_1^2 \Pi_{1z} + \frac{\partial}{\partial z} \nabla \cdot \Pi_1 \right) = \varepsilon_2 \left( k_2^2 \Pi_{2z} + \frac{\partial}{\partial z} \nabla \cdot \Pi_2 \right) \]  

(3'16)

and:

\[ \frac{\mathbf{D}_{1n}}{\mathbf{D}_{2n}} = \frac{\mathbf{D}_{1n}}{\mathbf{D}_{2n}} = \frac{\mu_1 \mathbf{H}_{1n}}{\mu_2 \mathbf{H}_{2n}} \]

\[ k_1^2 (\nabla \times \mathbf{\Pi}_1)_n = k_2^2 (\nabla \times \mathbf{\Pi}_2)_n \]

\[ k_1^2 \frac{\partial \Pi_{1x}}{\partial y} = k_2^2 \frac{\partial \Pi_{2x}}{\partial y} \]  

(3'17)

Novotny claims that the boundary conditions reduce to the continuity of \(k^2 \Pi_x, \nabla \cdot \Pi, k^2 \Pi_z,\) and \(k^2 (\partial \Pi_x / \partial z)\) across the boundary.\(^5\)

### 3.2.2 The Vertical Dipole in the Hertz Potential Description

“The field of a vertical dipole is rotationally symmetric, and it is sufficient to consider a one-component Hertz vector.”\(^6\) Let’s assume that the potential has only a \(z\) component which takes the following forms for the two media:

\[ \Pi_{1z} = \frac{p_z}{4\pi \varepsilon_1} \exp(ik_1 R_0) + \frac{p_z}{4\pi \varepsilon_1} \int_0^\infty dk_{||} J_0(k_{||} \rho) e^{ik_{1z}(z+h)} A_1(k_{||}) \]

\[ \Pi_{2z} = \frac{p_z}{4\pi \varepsilon_1} \int_0^\infty dk_{||} J_0(k_{||} \rho) e^{ik_{1z}h-ik_{2z}z} A_2(k_{||}) \]  

(3'18)

where \(R_0^2 = \rho^2 + (z-h)^2, \rho^2 = x^2 + y^2\) and \(p_z\) is the dipole strength.

Note that the first term in the incident Hertz potential is just the free dipole field and can be expanded with the help of the Sommerfeld identity as:

\[ \frac{p_z}{4\pi \varepsilon_1} \exp(ik_1 R_0) = \frac{ip_z}{4\pi \varepsilon_1} \int_0^\infty dk_{||} k_{||} J_0(k_{||} \rho) e^{ik_{1z}(h-z)} \]  

(3'19)

The boundary conditions become simply:

\[ \frac{\partial \Pi_{1z}}{\partial z} = \frac{\partial \Pi_{2z}}{\partial z} \]

\[ k_1^2 \Pi_{1z} = k_2^2 \Pi_{2z} \]  

(3'20)

\(^4\)Recall that there are four boundary conditions in total.

\(^5\)There are some dark points here. (For instance, why do we ignore the last two boundary conditions, and what happens to the differentiations with respect to \(y\)?)

\(^6\)From Novotny[1].
Upon substitution we obtain:

\[
\begin{align*}
  k_{||} + ik_{1z}A_1 &= -ik_{2z}A_2 \\
  ik_{1z}^2 \frac{k_{||}}{k_{1z}} + k_1^2 A_1 &= k_2^2 A_2
\end{align*}
\]  

Solving this system of equations we finally arrive at (recall that \(k_1^2/k_2^2 = \varepsilon_1/\varepsilon_2 = \epsilon_{rel}\)):

\[
\begin{align*}
  A_1(k_{||}) &= i k_{||} \left( k_2^2 k_{1z}^2 - k_1^2 k_{2z}^2 \right) k_{1z} / k_{2z}^2 + i k_{||} \left( k_{1z} - \epsilon_{rel} k_{2z}^2 \right) = i k_{||} \left( r(p) \right) \\
  A_2(k_{||}) &= \frac{k_2^2}{k_2^2} \left( A_1 + i k_{||} \right) = i \epsilon_{rel} \frac{k_{||}}{k_{1z}} \left( r(p) + 1 \right) = i \sqrt{\epsilon_{rel}} \frac{k_{||}}{k_{1z}} \left( t(p) \right)
\end{align*}
\]

Obviously, the numbers \(r(p)\) and \(t(p)\) are the Fresnel coefficients. Recall that the way we had defined the Fresnel coefficients was a little different. Actually the following relations obtain: \(r(p) = -r_{pp}\) and \(t(p) = t_{pp}\). \(^7\)

It is left as an exercise to the reader to check and see that, for the vertical dipole case, the electric fields calculated by Novotny and given in the appendix of his paper are the same as those calculated in the beginning of this chapter.

### 3.2.3 The Vector Potential

Recall the important relation between the vector potential and the polarization field, eq.(1’13):

\[
A(r) = \frac{i \omega \mu_0}{4\pi} \int d\mathbf{r}' \frac{e^{ik_1 |r-r'|}}{|r-r'|} \mathbf{P}(r')
\]  

For a single dipole this equation simply becomes:

\[
A(r) = \frac{i \omega \mu_0}{4\pi} e^{ik_1 |r-r'|} \mathbf{P}_j
\]

We also remind the reader of the Sommerfeld expansion, eq.(1’30):

\[
\frac{e^{ik_1 r}}{r} = i \int_0^\infty dk \frac{k_{||} J_0(k_{||} r) e^{ik_1 |z-h|}}{k_{||} k_{1z}}
\]

Combining this with eq.(3’24) yields:

\[
A(r) = -\frac{\omega \mu_0}{4\pi} \int_0^\infty dk \frac{k_{||} J_0(k_{||} r) e^{ik_1 |z-h|}}{k_{||} k_{1z}} \mathbf{P}_j
\]

Meanwhile keep in mind that the wave numbers now depend on the medium as well.

\(^7\)Novotny is once again wrong. His definition of \(t(p)\) is inconsistent with what he claims to be taken from Born & Wolf\(^{[4]}\). In the appendix of his paper he gives the correct form.
Chapter 4

Final Remarks

There is something wrong in the theory developed in the previous two chapters. Novotny\textsuperscript{[1]} uses (or at least claims to use) the Sommerfeld expansion, but the results that we get are quite different. In fact, for the vertical dipole, only the reflected field is the same as that found by Novotny. Apparently, something is still missing...

The difference between our approach and Novotny’s approach is that he introduces the reflection and transmission at the Hertz potential level, prior to calculating the electric fields. This means that the differentiations involved in the calculation of the electric field will bring about some extra terms that our derivations are missing. (This is true for both the Weyl and for the Sommerfeld expansions.) So, what we should actually do is to find the reflection and transmission properties of the vector potential (or the Hertz potential), and then apply the gradients to determine the fields.

But the theory developed by Novotny is not satisfactory (at least for me). It has some weak points (and some wrong points of minor importance). It has to be generalized and performed in a cleaner way (i.e. without tricks). That largely remains to be done.

The Fortran code is described in a following chapter. The code is based on the derivations of this report. It will have to be adjusted when the corrections are made. (I don’t think it will have to be rewritten. Actually there will be -should be- only relatively small changes in the final expressions for the electric fields in the Sommerfeld and Weyl expansions.)
Chapter 5

Results

5.1 Free Dipole Field

Figure 5.1: The square of the magnitude of the total free dipole electric field. (Note that this is proportional to the intensity of the field.)

The software generated from the theory will be described in the next chapter. Here we will only quote and discuss some results.

There are five possible ways of calculating the free dipole electric field: three analytical procedures corresponding to the spherical, Sommerfeld and Weyl expansions; and two numerical corresponding to the last two expansions. The analytical calculations are exactly the same. (This means that they are the same up to something like the 15th digit.) The numerical calculations though, depend on the accuracy which is determined by the number of iterations in the integrations. (It is redundant to point out that increasing the number of iterations not only increases the accuracy, but also the total time the program is run.) Fig. 5.1 shows a sample output of the spherical wave expansion, while Fig. 5.2 is a comparison of the numerical calculation based on the Weyl expansion with the spherical expansion. Increasing the number of iterations in the integration greatly increases the time, so one has to choose between time and accuracy.

Having only one dimensional integration, the Sommerfeld expansion is much faster than the Weyl expansion. Still, it is an important check to have both expansions agree. That is why the Weyl expansion should not be abandoned.
Figure 5’2: Contour plots related to Fig.(5’1). Comparison between the spherical and Weyl expansions. (The Sommerfeld expansion is more precise than the Weyl expansion.)

5.2 Reflected Field

5.3 Total Field in the Incidence Medium

5.4 Total Field in the Second Medium (Transmitted Field)

5.5 Checking for the Correctness of the Theory and Program

When the two media have the same permittivity we expect the reflected field to vanish, and the transmitted field to equal the free dipole field. This is indeed the case with the Sommerfeld expansion, as can be seen from Fig.(5’4). In this graph we have considered relatively distant points. For this case the impact of increasing the number of iterations is especially evident in the Weyl expansion, as can be seen from Fig.(5’5).
Free Dipole / Spherical Wave Expansion
Magnitude of the Poynting vector (i.e. the radiated field)
dipole position = (0,0,0); dipole moment = (1,0,0)

Figure 5′3: The square of the magnitude of the free dipole Poynting vector. (In a way, this is the intensity of the radiated field.) The dipole and the points where the calculations have been made are coplanar.

Convergence of the Sommerfeld Expansion
As the Number of Iterations Is Increased

Figure 5′4: As the number of iterations in the Sommerfeld expansion is increased from 5 to 10, the results become more and more precise. A lower number of iterations might be satisfactory for nearby points.
The Difference Between 5 and 6 Iterations in the Weyl Expansion

The Sommerfeld Expansion with 6 Iterations

The Weyl Expansion with 6 Iterations

(The Wild Oscillations at the Background Are the Weyl Expansion with 5 Iterations.)

Figure 5': The same situation as in Fig. (5′4), examined with the Weyl expansion. (It takes approximately 3 min to complete the 5 iterations, while this time increases to something like 30 min for 6 iterations. But the difference is loudly outspoken.)
Chapter 6

Numerics

6.1 The Structure

We have implemented all the formulas derived in this report into a Fortran code. The result was a moderately sized software which we are now to describe in a little detail.

We begin with examining the files and directories of the code. We provide a visual aid through fig.(6'1). The main directory which includes all of the software is Dipole_Surface/. It includes a makefile which compiles the code, a read_me with some information about the files and directories of the software, and several subdirectories, namely Figs/ (gnuplot set files and postscript graphs), Input/ (the input specifications), Obj/ (object files created by the makefile), Res/ (the output files), and Src/ (the source code). Additionally, the compiled executable file which is given the name execute is also created under this folder. Actually, some of the subdirectories here have subsubdirectories in themselves, but this is left as an exercise to the exploring reader...

All of the code is contained in the Src/ directory. The input data are written in a file input_data under the Input/ directory. The results are (by default) written into files ending with .res under directory Res/. The structure of this directory is arranged to aid our beloved reader (you): there are three directories named Free_Dipole/, Re
ection/ and Transmission/ intended for storing related data files, and a Notes/ directory including some information and comparisons regarding the timing and accuracy of the code (it might be useful to have a closer look at the files there).

The makefile runs four important commands. The first is the make command, which should be rather familiar to the computer using reader. Another is the make clean command which is intended to clean (remove) the object files created during the compilation under directory Obj/. The third and fourth commands are make plot2d and make plot3d. These invoke the gnuplot command files carrying the ending .set and located under directory Figs/Set/. The files in question are contour.set and surface.set. These create a contour and a surface plot of the results, provided, of course, that the results are three dimensional. The plots are in postscript format and are stored in the Figs/Ps/ directory under the names contour.ps and surface.ps, respectively. By default, the result data file should be renamed as output and left under the same directory, i.e. Res/. Similar to Res/, the Figs/Ps/ directory has three separate subfolders for the free dipole, reflected, and transmitted fields. Each of these is further divided up into Contour/ and Surface/ subdirectories, as shown in fig.(6'1).
Figure 6'1: Folders and files. (“D” stands for “Directory” and “F” stands for “File”.)

Figure 6'2: The rough hierarchy between the source files. (All files are in directory `Src/` unless otherwise indicated; names of major subroutines are given in parantheses.)
6.2 The Software

In this section we will have a closer look at the Fortran code contained in the Src/ folder.

![Diagram of code structure]

Figure 6.3: Calculation of the free dipole field. (“S” stands for “Subroutine”, “P” stands for “Program”, and “F” stands for “Function”. The name of the file where the piece of code is located is given in parantheses.)

6.3 Practical Problems and Difficulties

time...
accuracy, singularities...

6.4 One and Two Dimensional Integration

In the computer implementation of the formulas we derived we face a deep problem: integration. There seems to be no way to avoid it. (But we should not claim that there is certainly no way to find analytical solutions, or at least simplify a little more; actually it might be well worth it to invest some more time in searching a way around the integrations.) Recall that in the Sommerfeld expansion we have one dimensional, and in the Weyl expansion, two dimensional infinite complex integration. Although we could avoid singularities for the first,
Figure 6′4: Calculation of the reflected field. (See fig.(6′3) for explanations.) Calculation of the transmitted field has a very similar structure, and is left as an exercise to the structure-interested reader.

the second still has some badly-behaving parts. But there is no need to worry. It is possible to carry on the integrations numerically...

We first consider the one dimensional case. In our programs we use a mutated version of the “Numerical Recipes” subroutines midpnt and midinf. Conventionally we have renamed them cmidpnt and cmidinf, to imply explicitly that we have a double precision complex integration routine.

The two dimensional integration problem will be discussed in the section to follow, where we treat the more general three dimensional integration...

6.5 3-D Integration

To evaluate three dimensional real integrals over finite volumes we can use the simple procedure outlined in “Numerical Recipes”. Let’s now have a quick glimpse at how that looks like. There are two main subroutines: ggaus and quad3d. The first performs integration using the so-called Gaussian quadrature. The second is the subroutine that puts the computations into order; it needs three auxiliary functions $f$, $g$, and $h$ for each function to be integrated. The book suggests that three copies of the Gaussian quadrature routine be used, since in
Fortran a subroutine is unable to call itself. The three carbon copies are then named: *qgausx*, *qgausy*, and *qgausz*. The bounds of integration and the function are all user-defined; it is possible to define functional bounds as well. The idea behind the procedure is quite simple. Suppose one needs an integration of the form:

\[ I = \iiint f(x, y, z) \, dx \, dy \, dz \]

\[ = \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z) \, dz \]

(6'1)

All we do is to perform the integrations one by one. We make the following definitions which are implemented in exactly the same way into the computer program:

\[ F(x, y, z) = f(x, y, z) \]

\[ G(x, y) = \int_{z_1(x,y)}^{z_2(x,y)} F(x, y, z) \, dz \]

\[ H(x) = \int_{y_1(x)}^{y_2(x)} G(x, y) \, dy \]

\[ I = \int_{x_1}^{x_2} H(x) \, dx \]

(6'2)

In the program, the above functions are referred to as *Ffunc*, *Gfunc*, and *Hfunc*, respectively. One needs five user-defined functions: *Tfunc(x,y,z)*, *y1(x)*, *y2(x)*, *z1(x,y)*, and *z2(x,y)*. The first is the function to be integrated. The other four define the upper and lower y and z boundaries of the integration, as indicated in the above equations. The x boundary is defined explicitly in the main subroutine.

It is also possible to do two dimensional integration with the software. One just needs to set *z1 = 0* and *z2 = 1*. If the function is independent of z, then the integration along the z component will yield a mere 1. In view of this, the case for one dimensional integration is trivial...

However, our present needs are somewhat beyond the scope of “Numerical Recipes”: we need infinite two dimensional complex integration for the Weyl (plane wave) expansion. Yet, it is still possible to use a similarly structured algorithm; and that is what we do.

The “Numerical Recipes” routine *quad3d* has been extensively modified to *cinte3d*, now a complex integration routine. This routine uses another one dimensional integration routine *cmidpnt*, which is the complex version of the “Numerical Recipes” routine *midpnt*. The procedure is still to divide a 3D integration into three 1D integrations. A new integer parameter *nmax* has been defined and placed in a common block named *Times*; the parameter controls the number of iterations (or calls to the subroutine *cmidpnt*), which in turn, controls the number of points used in the evaluation of the integrals. The routine *cmidpnt*, which does the actual evaluations, triples the number of points on each subsequent call. In other words, the routine first evaluates the integral using one point only; on the second call, this is increased to three, and so on...

It is possible to perform the integration in cartesian or spherical coordinates. Only slight modifications are required; yet the outputs are rather different. As a test and comparison, we have evaluated the following integral:

\[ \int_V e^r \, dv = \int_{-1}^{1} \int_{0}^{\pi} \int_{0}^{2\pi} e^r r^2 \sin \theta \, dr \, d\theta \, d\phi = 4\pi(e - 1) \]

(6'3)

where the integration is performed over a sphere of radius 1 (unit sphere), centered at the origin. The results follow (only the real part is considered):
integration type: cartesian spherical
real value : 9.02619566197593 9.02619566197593
nmax=1 : 8.00000000000000 8.136135473581
nmax=2 : 10.0593547885949 8.96114376301536
nmax=3 : 9.20660198704750 9.01927024689608
nmax=4 : 9.05721536999079 9.02542978899295
nmax=5 : 9.03170811005298 9.02611060948218
nmax=6 : 9.02720330569999079 9.02619461201295
nmax=7 : 9.02638351736272 9.02619461201295
nmax=8 : 9.02623113104342 ...
(takes some time)
(takes quite some time)

The interpretation of the results is left as an exercise to the reader...

At this point, it may be useful to give the user-defined parts of the program to evaluate the above results (in cartesian coordinates only; it is pointless to repeat that the rest is left as an exercise):

\[
\begin{align*}
  f(x, y, z) &= \exp \left( \sqrt{x^2 + y^2 + z^2} \right) \\
  y_1(x) &= \sqrt{1 - x^2} \\
  y_2(x) &= -\sqrt{1 - x^2} \\
  z_1(x, y) &= \sqrt{1 - x^2 - y^2} \\
  z_2(x, y) &= -\sqrt{1 - x^2 - y^2}
\end{align*}
\]

(6'4)

6.6 To Do (or not to Do)

The code can be improved further...
Appendix A

Bessel Functions

Definitions

There are many different types of Bessel functions. But in the text we made use of only two of them. These are the so-called Bessel functions of the first kind of integer order, \( J_\nu \) and modified Bessel functions of the first kind of integer order, \( I_\nu \). The subscript \( \nu \) is an integer denoting the order of the function.

Here we quote the definitions given in *Abramowitz & Stegun*, eqs.(9.1.10) and (9.6.10) respectively:

\[
J_\nu(z) = \left( \frac{1}{2}z \right)^\nu \sum_{k=0}^{\infty} \frac{\left( -\frac{1}{4}z^2 \right)^k}{k!\Gamma(\nu + k + 1)} \quad (A'1)
\]

\[
I_\nu(z) = \left( \frac{1}{2}z \right)^\nu \sum_{k=0}^{\infty} \frac{\left( \frac{1}{4}z^2 \right)^k}{k!\Gamma(\nu + k + 1)} \quad (A'2)
\]

The Gamma function, \( \Gamma \) is defined by eq.(6.1.1) (this was previously quoted in the text):

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad (A'3)
\]

Fig.(A'1) plots the first three members of the \( J_\nu \) family.
Important Relations and Other Formulas

Here we shall list some important relations which were of use.

a) Eq.(9.1.27); used in the derivation of the cylindrical expansion; then quoted as eqs.(1’32) and (1’39), respectively:

\[
\frac{d}{dz} J_\nu(z) = -J_{\nu+1}(z) + \frac{\nu}{z} J_\nu(z) \quad (A'4)
\]
\[
2\nu J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z) \quad (A'5)
\]

b) Eq.(9.6.3); used in the section comparing the Sommerfeld and Weyl expansions; then quoted as eq.(1’85):

\[
I_\nu(z) = e^{-\nu\pi i/2} J_\nu(ze^{\pi i/2}) \quad (-\pi < \arg z \leq \pi/2) \quad (A'6)
\]
\[
I_\nu(z) = e^{3\nu\pi i/2} J_\nu(ze^{-3\pi i/2}) \quad (\pi/2 < \arg z \leq \pi) \quad (A'7)
\]

c) Eq.(11.4.17); used in the section describing some special cases related to the Sommerfeld expansion; then quoted as eq.(1’72):

\[
\int_0^\infty J_\nu(t)dt = 1 \quad (A'8)
\]

d) Eq.(11.4.16); used alongside the above equation; then quoted as eq.(1’73):

\[
\int_0^\infty t^\mu J_\nu(t)dt = \frac{2^\nu \Gamma \left(\frac{\nu+\mu+1}{2}\right)}{\Gamma \left(\frac{\nu-\mu+1}{2}\right)} \quad (A'9)
\]

e) Eq.(9.6.16); used to prove the relation between the Sommerfeld and Weyl expansions; quoted in the text as eq.(1’84):

\[
I_0(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} d\theta \quad (A'10)
\]
Bibliography


